Dr. Stephan Juricke, Prof. Sören Petrat (based on lecture notes by Marcel Oliver)
Jacobs University, Fall 2022

4. Differential Equations

Topic 4.A: Common Ordinary Differential Equations

In many applications, we are given a relation between functions and their derivatives, e.g.,

Newton's eq. $m\dfrac{d^2x(t)}{dt^2} = F(x(t))$.

Here, let us discuss the case where only the first derivative is involved.

Definition:

For some given function f, we call $\dfrac{dy}{dt} = f(y(t),t)$ a first-order **ordinary differential equation (ODE)**.

If $\dfrac{dy}{dt} = f(y(t))$ (no explicit t dependence of f) we call the ODE **autonomous.**

If $\dfrac{dy}{dt} = f(y(t))\,g(t)$ for some fct. g we call the ODE **separable**.

Goal: Find functions $y(t)$ that satisfy $\dfrac{dy}{dt} = f(y(t),t)$, for given f.

A very important solution technique that works for separable ODEs is:

Solution technique: Separation of variables

For $\dfrac{dy}{dt} = f(y)g(t)$, a solution can be found by integration: $\displaystyle\int_{y_0}^{y(t)} \frac{1}{f(y)}\,dy = \int_{t_0}^{t} g(x)\,dx$.

Note: • This works if f and g are continuous and $f(y) \neq 0$ for $y \in [y_0, y(t)]$.

$\hookrightarrow y_0 = y(t_0) =$ initial condition

• Formally: Bring all y's to one side, all t's to the other, and then integrate.

$\left(\text{"} \frac{dy}{dt} = f(y) g(t) \Rightarrow \frac{dy}{f(y)} = g(t)\,dt \Rightarrow \int \frac{dy}{f(y)} = \int g(t)\,dt \text{"} \right)$

Next: a few important examples

Exponential Growth:

$$\underbrace{\boxed{\frac{dy}{dt} = \lambda y}}_{\text{rate of change of } y \text{ is proportional to } y}$$, with parameter $\lambda \in \mathbb{R}$ (separable and autonomous: $f(y) = \lambda y$, $g(t) = 1$)

Examples: • epidemic: $\lambda = \beta - \gamma$

rate coefficient for recovery (γ)

transmission rate coefficient (β)

$y(0) = y_0 =$ initial number of infections

For $\lambda > 0$: number of infections is increasing,

$\lambda < 0$: number of infections is decreasing.

• radioactive decay: $\lambda < 0$

• generally: unrestricted population growth (for $\lambda > 0$) in this model

Solution: $\frac{dy}{dt} = \lambda y \Rightarrow \underbrace{\int_{y_0}^{y(t)} \frac{1}{y}\,dy}_{= \ln y \big|_{y_0}^{y(t)} = \ln y(t) - \ln y_0 = \ln \frac{y(t)}{y_0}} = \lambda \underbrace{\int_0^t dt}_{} = \lambda t$ (assume $y_0 > 0$)

$$\Rightarrow \ln \frac{y(t)}{y_0} = \lambda t \Rightarrow \frac{y(t)}{y_0} = e^{\lambda t} \Rightarrow y(t) = y_0 e^{\lambda t} \quad (\text{with } y(0) = y_0).$$

Note: Generally one (integration) constant (here y_0) is determined by initial conditions.

Note: Doubling time T_2 = time it takes for y to double:

$$y(T_2) = y_0 e^{\lambda T_2} = 2y_0 \implies 2 = e^{\lambda T_2} \implies T_2 = \frac{\ln 2}{\lambda}.$$

(If $\lambda < 0$, we speak of "half-life", e.g., radioactive decay.)

Limited Growth:

Growth might be limited, e.g., by limited food supply (e.g., fish in a pond).

$\implies$ Growth should stop once $y = k$ is reached, k = "environmental carrying capacity".

$$\implies \boxed{\frac{dy}{dt} = \underbrace{\lambda y}_{\text{exp. growth}} \underbrace{\left(1 - \frac{y}{k}\right)}_{\text{stopped if } y=k}}$$

(this is called "logistics equation")

Separation of variables: $\displaystyle\int_{y_0}^{y(t)} \underbrace{\frac{1}{y(k-y)}} dy = \frac{\lambda}{k} \int_0^t dx = \frac{\lambda}{k} t$

recall integration of rational functions

$$= \frac{A}{y} + \frac{B}{k-y} = \frac{A(k-y) + By}{y(k-y)} \quad \text{(partial fractions)}$$

$$\implies B - A = 0 \implies A = B \text{ and } Ak = 1 \implies A = B = \frac{1}{k}$$

$$\implies \int_{y_0}^{y(t)} \frac{1}{y(k-y)} dy = \frac{1}{k} \left(\int_{y_0}^{y(t)} \frac{1}{y} dy + \int_{y_0}^{y(t)} \frac{1}{k-y} dy \right)$$

$$= \frac{1}{k} \left(\ln y \Big|_{y_0}^{y(t)} - \ln(k-y) \Big|_{y_0}^{y(t)} \right) = \frac{1}{k} \ln \frac{y}{k-y} \Big|_{y_0}^{y(t)}$$

$$= \frac{1}{k} \left(\ln \frac{y(t)}{k - y(t)} - \ln \frac{y_0}{k - y_0} \right) = \frac{1}{k} \ln \left(\frac{y(t)}{k - y(t)} \, \frac{k - y_0}{y_0} \right)$$

$\Rightarrow$ Solution is given by $\ln\left(\frac{y(t)}{k-y(t)}\,\frac{k-y_0}{y_0}\right) = \lambda t$

$$\Rightarrow \frac{y(t)}{k-y(t)} = \frac{y_0}{k-y_0}\, e^{\lambda t} =: \alpha$$

Still need to solve $\frac{y}{k-y} = \alpha \Rightarrow y = \alpha(k-y) = \alpha k - \alpha y \Rightarrow (1+\alpha)y = \alpha k$

$$\Rightarrow y = \frac{\alpha}{1+\alpha}\, k = \frac{k}{\frac{1}{\alpha}+1}$$

$$\Rightarrow \boxed{y(t) = \frac{k}{\frac{(k-y_0)}{y_0}e^{-\lambda t}+1} = \frac{k y_0}{(k-y_0)e^{-\lambda t}+y_0}} \qquad (\text{for } y_0 \geq 0)$$

Note: Initial condition: $y(t=0) = \frac{k y_0}{(k-y_0)+y_0} = y_0$ ✓

We could now plot $y(t)$ to discuss how it behaves for different initial data.

But even without an exact solution we can discuss the qualitative behavior.

Recall our ODE $\frac{dy}{dt} = \lambda y\left(1-\frac{y}{k}\right)$:

- for $y_0 = 0$: $\frac{dy}{dt} = 0 \Rightarrow y(t) = 0 \quad \forall t \geq 0$

  (check with exact solution: $y(t) = \frac{k\cdot 0}{(k-0)e^{-\lambda t}+0} = 0$ ✓)

- for $y_0 = k$: $\frac{dy}{dt} = 0 \Rightarrow y(t) = k \quad \forall t \geq 0$

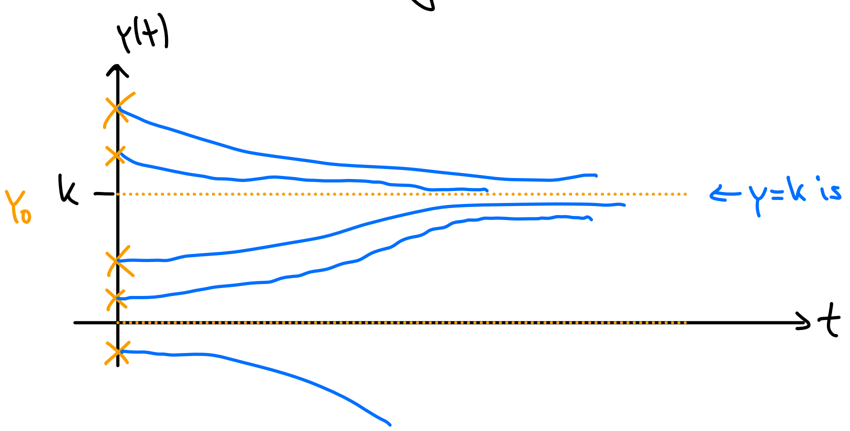
  (check with exact solution: $y(t) = \frac{k\cdot k}{(k-k)e^{-\lambda t}+k} = k$ ✓)

These points where $\frac{dy}{dt} = 0$ are called $\boxed{\text{equilibrium points.}}$

Then: for large t:

- when $y_0 > k$: $\frac{dy}{dt} < 0$, i.e., y is decreasing — $y(t) \to k$
- when $y_0 \in (0,k)$: $\frac{dy}{dt} > 0$, i.e., y is increasing — $y(t) \to k$
- when $y_0 < 0$: $\frac{dy}{dt} < 0$, i.e., y is decreasing — $y(t) \to -\infty$

This yields the following qualitative sketch:



← $y=k$ is an attractive or stable equilibrium point

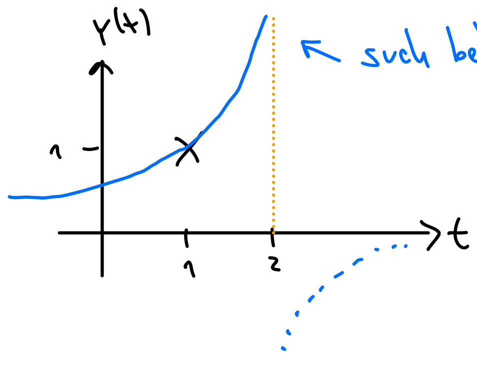
Note: Here, the solution is unique, so two solution curves cannot cross!

Two more examples:

- $\frac{dy}{dt} = y^2$ with initial condition $y(1) = 1$

$$\Rightarrow \int_1^{y(t)} \frac{1}{y^2}\, dy = \int_1^t dx \Rightarrow -\frac{1}{y}\Big|_1^{y(t)} = x\Big|_1^t \Rightarrow -\frac{1}{y(t)} + 1 = t-1$$

$$\Rightarrow y(t) = \frac{1}{2-t}$$



↖ such behavior is called "blow-up" at $t=2$

Such blow-ups can appear if LHS of separation of variables has singularities (here, $\frac{1}{y^2}$).

- $\frac{dy}{dt} = \sqrt{y}$ with $y(0) = y_0$

$\Rightarrow \int_{y_0}^{y(t)} \frac{1}{\sqrt{y}}\, dy = \int_0^t dx \Rightarrow 2y^{\frac{1}{2}}\Big|_{y_0}^{y(t)} = t \Rightarrow \sqrt{y(t)} = \frac{t}{2} + \sqrt{y_0}$

$\Rightarrow y(t) = \left(\frac{t}{2} + \sqrt{y_0}\right)^2$

E.g., if $y_0 = 0$: $y(t) = \frac{t^2}{4}$

But: $y(t) = 0$ is also a solution.

In fact, for any $c > 0$, $y(t) = \begin{cases} 0 & \text{for } t < c \\ \frac{(t-c)^2}{4} & \text{for } t \geq c \end{cases}$ is also a solution (for $y(0) = 0$)

So here the solution for the same initial condition is not unique.

(Reason is again the singularity of $\frac{1}{\sqrt{y}}$.)