Calculus and Elements of Linear Algebra I Session 2

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1. Functions

1.1 Numbers and Polynomials

Topic 1.1.B: Complex Numbers and the Fundamental Theorem of Algebra

Last time we noted: If the discriminant of a quadratic eq. is negative, there is no real solution.

Example: $x^2 = -1$.

Solution: We define the $\boxed{\text{imaginary unit } i}$ with the property $\boxed{i^2 = -1}$.

With this we define the $\boxed{\text{complex numbers } \mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}}$.

If $z = x + iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$, we call

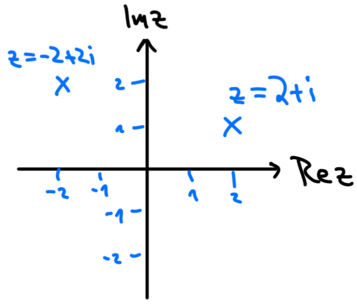
- $x = \operatorname{Re} z$ the real part of z,
- $y = \operatorname{Im} z$ the imaginary part of z,
- $\underbrace{z^*}_{\text{written } \bar{z} \text{ sometimes}} = x - iy$ the complex conjugate of z.

If $\operatorname{Re} z = 0$, i.e., $z = iy$, we call $z = iy$ purely imaginary.

Example: Multiplication of complex numbers:

$$(2+3i)(4-i) = 2 \cdot 4 + 2 \cdot (-i) + 3i \cdot 4 + 3i \cdot (-i) = 8 - 2i + 12i - 3\overbrace{i^2}^{=-1} = 11 + 10i.$$

Note: We can draw a complex number in the $\underbrace{\text{complex plane}}_{\text{also called Argand plane}}$:



Now we have all the numbers we need: $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.

(Note on notation: $\mathbb{R} \subset \mathbb{C}$ = "$\mathbb{R}$ is a subset of $\mathbb{C}$" $\Longleftrightarrow$ $\mathbb{C} \supset \mathbb{R}$ = "$\mathbb{C}$ is a superset of $\mathbb{R}$".)

Now back to polynomials:

Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial with real coefficients $a_0, \ldots, a_n$.

Suppose z is a root, i.e., $p(z) = 0$.

Then: $p(z^*) = a_n \underbrace{(z^*)^n}_{\text{we will show in the HW that } (z^*)^n = (z^n)^*} + a_{n-1}(z^*)^{n-1} + \ldots + a_1 z^* + a_0$

$$\Rightarrow p(z^*) = a_n (z^n)^* + a_{n-1}(z^{n-1})^* + \ldots + a_1 z^* + a_0$$

$$= \left(a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0\right)^* \quad \Big\} \text{ because } (z_1 + z_2)^* = z_1^* + z_2^* \text{ and } a z^* = (az)^* \text{ for } a \in \mathbb{R}$$

$$\overset{z \text{ is a root}}{=} 0^* = 0.$$

We conclude: If z is a root, so is z^*. (We call z and z^* a complex conjugate pair.)

Let us come back to our discussion of roots of polynomials of degree n:

- $n = 1$: always one real root
- $n = 2$: $a_2 x^2 + a_1 x + a_0 = 0$. If discriminant $\Delta = a_1^2 - 4a_2 a_0$ $\begin{cases} > 0: \text{ two distinct real roots} \\ = 0: \text{ one "double" real root} \\ < 0: \text{ complex conjugate pair of roots} \end{cases}$
- $n = 3, 4$: formulas exist, but very lengthy
- $n \geq 5$: no general formula exists! (Abel-Ruffini theorem)

But, in general, we always know the following:

Fundamental Theorem of Algebra:

Every polynomial of degree $n \geq 1$ has at least one complex root.

From this we can deduce:

Corollary:

Every polynomial of degree $n \geq 1$ has n complex roots $z_1, \dots, z_n$ (not necessarily distinct), and can be written in factorized form as $p(x) = a_n (x-z_1)(x-z_2)\cdots(x-z_n)$.

Note: If a root appears k times, we say it is a root of order k, e.g., a "double root" or "triple root" etc.

Idea of proof: We use polynomial long division (which we discuss next time) to write

$$p(x) = (x-\alpha)q(x) + r \quad , \text{with } \alpha, r \in \mathbb{C}, \text{ and } q(x) \text{ a polynomial of degree } n-1.$$

Now: If α is a root of $p(x)$, then $0 = p(\alpha) = \underbrace{(\alpha-\alpha)q(\alpha)}_{=0} + r = r$, i.e., $r = 0$, i.e.,

$p(x) = (x-\alpha)q(x)$. Then repeat with $q(x)$.