Calculus and Elements of Linear Algebra I Session 20

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

5. Vectors & vector spaces

Topic 5.A: Introduction to vectors & vector operations

Vectors in Euclidean space $\mathbb{R}^n$, with $n = 2,3$ mostly (i.e. common physical 2D and 3D space)

- A quantity with magnitude and direction
- Can be thought of specifying a position (displacement of origin) or displacement, but generally not both at the same time
- Is represented by coordinates

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \leftarrow \text{components or entries}$$

↳ column vector (default form of vector)

- Notation: $\vec{v}$, $\underline{v}$, boldface $\mathbf{v}$

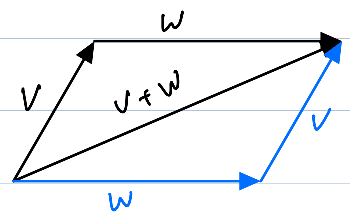
  In the following: no special indication!

  (should be clear from context)

- Two basic operations:

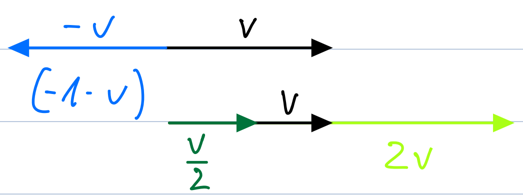
  1) Addition / Subtraction

$$\boxed{v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}}$$



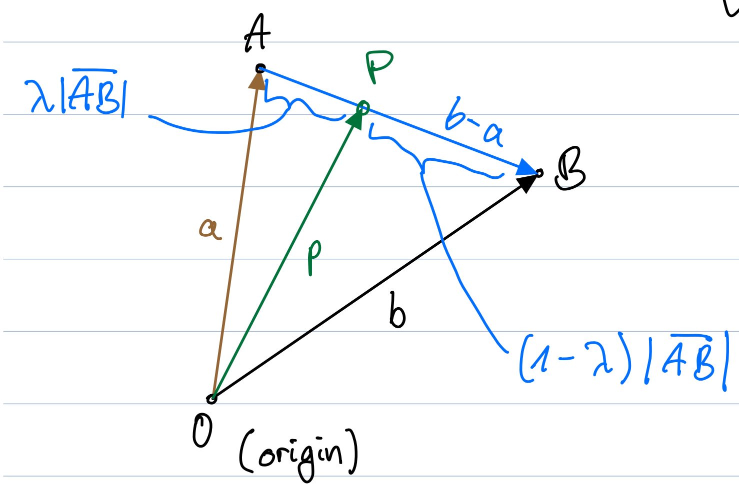
  2) Scalar multiplication

$$\boxed{\lambda v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}}$$ with scalar $\lambda \in \mathbb{R}$ (or even $\mathbb{C}$)



These operations obey the usual arithmetic rules. More details will follow for vector spaces

Ex.: Point on a line segment



$a + (b-a) = b$

Suppose P is located a fraction λ of the length of the line segment from A to B ($|\overline{AB}| > 0$).

$$\Rightarrow \quad p = a + \lambda(b-a) = (1-\lambda)a + \lambda b$$

for $\lambda \in [0,1]$

This is called a convex combination, as the non-negative coefficients λ and $(1-\lambda)$ sum to 1.

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Magnitude or length of a vector:

$$\boxed{|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}} \quad \text{for } v \in \mathbb{R}^3$$

Unit vector, i.e. with length 1, in direction of $v \neq 0$:

$$\hat{v} = \frac{v}{|v|} \Rightarrow |\hat{v}| = 1 \quad \text{(other notations possible)}$$

It encodes only direction information

Polar decomposition: $v = \underbrace{|v|}_{\text{length (scalar)}} \underbrace{\hat{v}}_{\text{direction}}$

Scalar product or **inner product** or **dot product**:

$$\boxed{u \bullet v = |u|\,|v| \cos\theta}$$

, with θ as angle between u and v

both are vectors, so not "normal" multiplication

The $\bullet$ is often printed larger than for "normal" multiplication between two scalars or a vector and a scalar, for which multiplication signs are often not written at all, e.g. λv, $5x$ etc.

When two vectors are involved, $\bullet$ means scalar product.

Also:

$$\boxed{\begin{aligned} u \bullet v &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= u^T v \end{aligned}}$$

for $u, v \in \mathbb{R}^3$

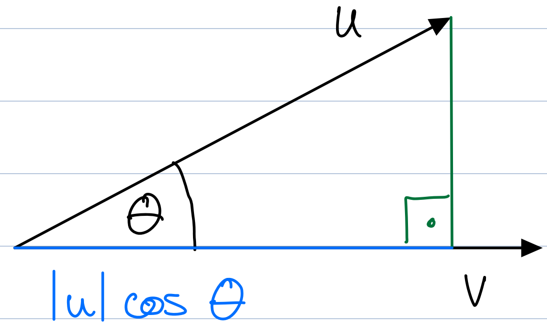
where u^T is row vector (u_1, u_2, u_3) and ^{T} stands for transpose.

We'll discuss this later.

Note: $u \cdot v = 0$, then u is perpendicular to v, $u \perp v$, as $\cos 90° = \cos 270° = 0$

(with the understanding that $\vec{0} \perp \vec{v}$ for any $v \in \mathbb{R}^n$)

Sketch:

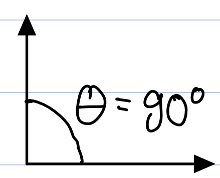


$|u| \cos \theta$

projection of u onto v

$\Rightarrow |u||v| \cos \theta$ is the scaled projection of u onto v (or other way around)

If



no vector projection > 0

Remark: If u, v have complex entries, then

$$\boxed{\begin{aligned} u \cdot v &= u_1^* v_1 + u_2^* v_2 + u_3^* v_3 \\ &= u^H v \end{aligned}} \quad u, v \in \mathbb{C}^3$$

where u^H is the complex conjugate transpose and ^{H} stands for Hermitian.

v

We then have $u \circ v = (v \circ u)^*$

In all cases: $\boxed{|u|^2 = u \bullet u}$

always $\in \mathbb{R}$ since $u \cdot u = (u \cdot u)^*$
for $u \in \mathbb{C}^n$ and therefore $u \cdot u \in \mathbb{R}$

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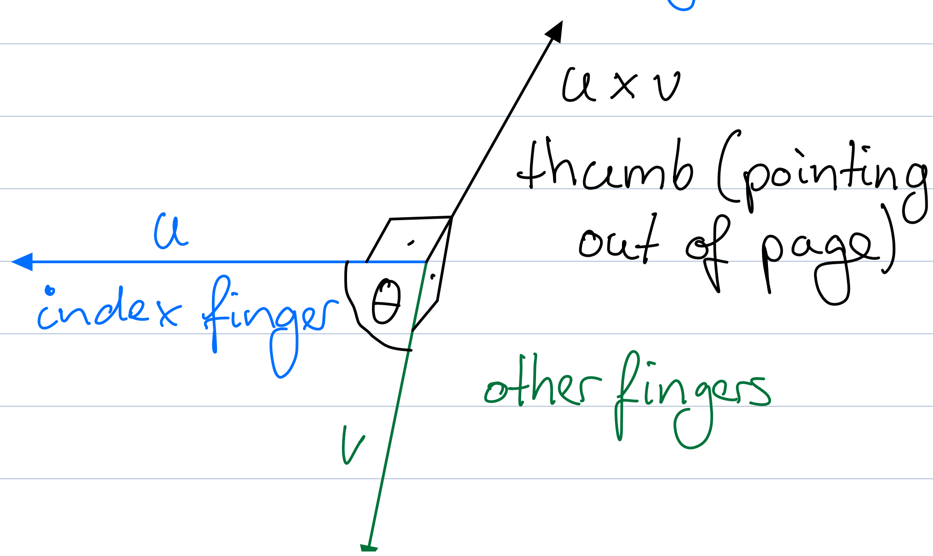
Cross product in $\mathbb{R}^3$:

$$|u \times v| = |u|\,|v| \sin\theta$$

in the direction that is perpendicular to both u and v with the convention that

$u \times v$, u, and v are a right-handed system



using fingers of your right hand

In a left-handed system, the vector (the thumb) would point into rather than out of the page.

Coordinate expression:

$$\boxed{u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}} \quad \begin{array}{l} \leftarrow \text{index } 2,3 - 3,2 \\ \leftarrow \text{index } 3,1 - 1,3 \\ \leftarrow \text{index } 1,2 - 2,1 \end{array}$$

again, when using two vectors

One way to memorize:

one shifted up from 1,2,3

one more shifted up 2,3,1

$$\begin{array}{cccc} 2 & 3 & - & 3 \;\; 2 \\ 3 & 1 & - & 1 \;\; 3 \\ 1 & 2 & - & 2 \;\; 1 \end{array}$$

positions u

positions v

vice versa indices

The outcome is a vector rather than a scalar as was the case for the scalar product.

If u and v are parallel, i.e. $\Theta = 0°$ or $180°$, $\sin\Theta = 0$, and so the length of $u \times v$ is zero and $u \times v = \vec{0}$ the zero vector (not just the number 0).

Properties:

- $(u+v) \times w = u \times w + v \times w$
- $u \times v = -v \times u$
- $u \times (v \times w) \neq (u \times v) \times w$