Calculus and Elements of Linear Algebra I Session 21

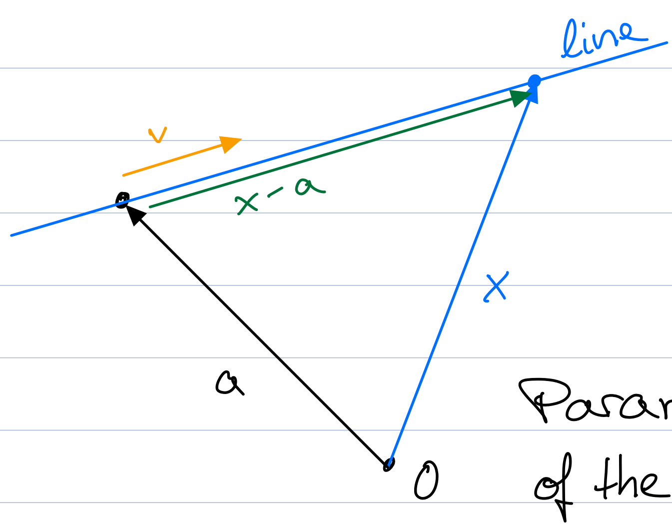
Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

5. Vectors & vector spaces

Topic 5.B: Lines, planes and vector spaces

Equation of line



vector of direction of line

$x = a + \lambda v, \quad \lambda \in \mathbb{R}$

vector of point on line

Parametric representation of the line with parameter λ

Ex.: Find the equation of the line through a and b:

Take a as reference point
Take direction vector $v = b - a$ (amplitude of vector does not matter, as long as $v \neq \vec{0}$)

$$\Rightarrow \quad x = a + \lambda (b - a)$$

This is not a unique line representation. Other choices are possible, e.g. switch b and a. We just need <u>some</u> point on the line and <u>some</u> vector in the direction of the line.

<u>Note:</u> Take the vectors a & b of points on the line, the vector v in the direction of the line. x is on the line if

$$\boxed{x - a \parallel v}$$

$\uparrow$ parallel to

$$\Leftrightarrow (x - a) \parallel (a - b)$$
$$\Leftrightarrow (x - a) \times (a - b) = 0$$

$\uparrow$ cross product

(cross product $= 0$ if vectors are parallel, and max. if vectors are perpendicular)

Calculus and Elements of Linear Algebra I Session 21

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

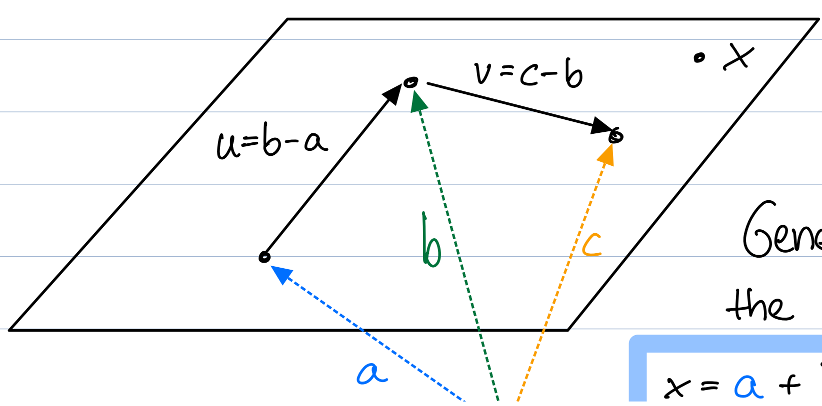
5. Vectors & vector spaces

Topic 5.B: Lines, planes and vector spaces

Equations of planes

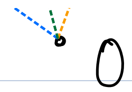
Parametric equation of the plane

Any set of two (independent, i.e. in this case non-parallel) linear equations in three dimensions defines a plane.



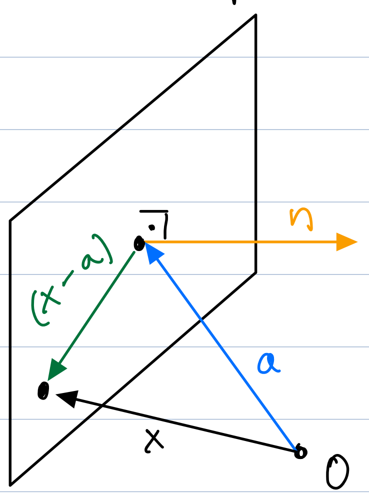
General point on the plane:

$$\boxed{x = a + \lambda u + \mu v \ , \ \lambda, \mu \in \mathbb{R}}$$



$\boxed{\text{Normal equation of a plane}}$

Normal vector to a plane is a vector pointing away from the plane in a right angle.



For normal vector

$(x-a) \perp n$ for any x on plane and some a on plane

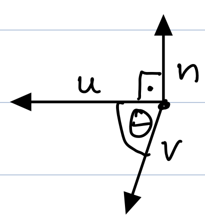
$\Leftrightarrow \boxed{(x-a) \cdot n = 0}$

Conversion of equations from one form to the other

1) From parametric representation to normal representation

We have $x = a + \lambda u + \mu v$

Set $n = u \times v$



so we get

$$(x-a) \cdot n = 0$$

2) From a general plane equation to normal representation

An equation of a plane has the general form (as extension of equation of a line)

$$\boxed{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta = \alpha \cdot x}$$

with some fixed coefficients $\alpha_1, \alpha_2, \alpha_3$ given by the vector
$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \in \mathbb{R}^3$$

and the fixed scalar $\beta \in \mathbb{R}$, s.t. all vectors
$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$
fulfilling the equation are pointing to points on the plane.

We take $n = \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$ and pick a point a s.t. $\beta = n \cdot a$, e.g. $a = \beta \frac{n}{|n|^2}$

$$\Rightarrow a \cdot n = \beta \frac{n}{|n|^2} \cdot n = \beta \underbrace{\cancel{\frac{\overbrace{n \cdot n}^{=|n|^2}}{|n|^2}}}_{=1} = \beta$$

We can now show that $n \perp (x-a)$, i.e.

$n \circ (x-a) = 0$ for any x that follows

the equation ⊛ :

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \bullet \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ x_3 - a_3 \end{pmatrix} = \alpha_1 (x_1 - a_1) + \alpha_2 (x_2 - a_2) + \alpha_3 (x_3 - a_3)$$

$$= \underbrace{\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3}_{= \beta \text{ since } x \text{ is on plane}} - \underbrace{(\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3)}_{= \beta \text{ by construction of } a}$$

$= \beta - \beta = 0$ so $n \circ (x-a) = 0$

We call $n \circ (x-a) = 0$ or $n \circ x = \beta$ with corresponding β and a also the point-normal form of the plane.

<u>Ex.:</u> $x_1 + 3x_2 - x_3 = 5$

$\Rightarrow$ normal form $n \cdot x = 5$ with $n = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

5. Vectors & vector spaces

Topic 5.B: Lines, planes and vector spaces

Vector spaces

A vector space is a abstract set of vectors that follows certain rules.

V is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) if

(i) V is closed under commutative and associative addition, i.e.:

$$+ : V \times V \to V$$

with
$$\left.\begin{aligned} a + b &= b + a \in V \\ (a+b)+c &= a+(b+c) \in V \end{aligned}\right\} \forall a,b,c \in V$$

(addition of vectors from V keeps them in V)

(ii) V is closed under distributive and associative scalar multiplication, i.e.:

$$\cdot : \underset{(\text{or } \mathbb{C})}{\mathbb{R}} \times V \to V$$

with
$$\left.\begin{aligned} \lambda\cdot(a+b) &= \lambda a + \lambda b \in V \\ (\lambda+\mu)\cdot a &= \lambda a + \mu a \in V \\ \lambda\cdot(\mu\cdot a) &= (\lambda\cdot\mu)\cdot a \in V \end{aligned}\right\} \begin{aligned} &\forall a,b \in V \\ &\forall \mu,\lambda \in \mathbb{R} \\ &(\text{or } \mathbb{C}) \end{aligned}$$

(multiplication of vectors from V with scalars from $\mathbb{R}$ (or $\mathbb{C}$, if vector space is over $\mathbb{C}$) keeps them in V)

(iii) $\exists\ 0 \in V: \quad a + 0 = a \quad \forall a \in V$
(neutral element of addition)

(iv) $1\cdot a = a \quad \forall a \in V$
($1 \in \mathbb{R}$ (and $\in \mathbb{C}$) is neutral scalar of scalar multiplication)

<u>Notes:</u>

- 0 is unique: Assume we have two different 0, 0 and $\mathbf{0}$

$$0 + \mathbf{0} = 0 \quad \Rightarrow \quad 0 = \mathbf{0}$$

$= 0 + 0 = 0$ are the same

$\lightning$

- For every $a \in U$ there is $-a := (-1) \cdot a$

$\Rightarrow a - a = 0$

Indeed: $a - a = \underbrace{(1-1)}_{=0} \cdot a = 0 \cdot a = 0$

<u>Examples:</u>

- $U = \mathbb{R}^n$, $n \in \mathbb{N}$
- $U = \mathbb{C}^n$, $n \in \mathbb{N}$
- $U = \{ p : p$ polynomials of degree $\leq n \}$

<u>Non-example:</u>

- $U = \{ p : p$ polynomials of degree $= n \}$

(has no neutral element of addition)

Calculus and Elements of Linear Algebra I Session 21

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

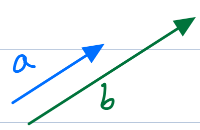
5. Vectors & vector spaces

Topic 5.B: Lines, planes and vector spaces

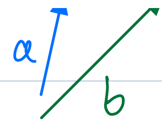
Def.: The vectors in $\{v_1, \ldots, v_m\}$ are linearly independent if

$$\underbrace{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_m v_m}_{\text{linear combination of } v_i,\ i=1,\ldots,m} = 0 \quad \text{implies}$$

$$\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0 \;!$$



linearly dependent
(a is multiple of b)



linearly independent

Def.: The set $\{b_1, \ldots, b_n\}$ is called a basis of V if

(i) For every $x \in V$ there are $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ s.t. $x = \alpha_1 b_1 + \ldots + \alpha_n b_n$

(ii) This representation is unique.

If $\mathcal{B}$ is a basis of V, the vector $\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \in \mathbb{R}^n$ is called the coordinate vector of x (given the basis $\mathcal{B}$).

We can also write
$$x = \sum_{i=1}^{m} \alpha_i b_i \quad \text{or simply}$$
$$= \sum_i \alpha_i b_i$$
if the upper limit is obvious from context.

Condition (i) of a basis can be written as

$$\forall x \in V \ \exists \alpha_1, \ldots, \alpha_n \ \text{s.t.} \ x = \sum_i \alpha_i b_i$$

or: $\boxed{V = \text{span}\{b_1, \ldots, b_n\}}$

set of all linear combinations of the indicated vectors

$\boxed{\text{Theorem: } \mathcal{B} \text{ is a basis} \Rightarrow \text{all vectors in } \mathcal{B} \text{ are linearly independent}}$

Fact: If V has basis $\mathcal{B}$ that is finite (i.e. finite number of vectors) than every other basis has the same number of vectors n.
We say that the dimension of V is n or
$\dim V = n$