Calculus and Elements of Linear Algebra I

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6. Matrices

6.1 Introduction to matrices and link to linear operators

6.1. A: Linear operations (transformations) on vector spaces

Def.: We call the operator $\mathcal{A}: V \to V$ *linear* if

(i) $\mathcal{A}(v+u) = \mathcal{A}v + \mathcal{A}u \quad \forall v, u \in V$

(ii) $\mathcal{A}(\lambda v) = \lambda \mathcal{A}(v) \quad \forall \lambda \in \mathbb{R}$ (or $\mathbb{C}$), $v \in V$

Note: The line equation $f(x) = y = ax + b$ is only linear in this sense if $b = 0$.
For example, taking $b \neq 0$:

$f(\lambda x) = a\lambda x + b \neq a\lambda x + \lambda b = \lambda(ax+b) = \lambda f(x)$

In case of $b \neq 0$ we say affine linear

Basis representation:

Take a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V, $v \in V$ with

$$v = \sum_j \alpha_j b_j \quad , \quad \alpha_j \in \mathbb{R}, \ b_j \in V$$

$$\Rightarrow \quad \mathcal{A} v = \mathcal{A}\Big(\sum_j \alpha_j b_j\Big) \quad \in V$$

$$(*) \qquad = \sum_j \alpha_j \mathcal{A} b_j \qquad \text{(using definition)}$$

Now suppose, since $\mathcal{A}v = w \in V$,

$$(**) \qquad \mathcal{A}v = \sum_i \beta_i b_i \quad , \quad \beta_i \in \mathbb{R}$$

and also, since $\mathcal{A} b_j = c_j \in V$,

$$(***) \qquad \mathcal{A} b_j = \sum_i a_{ij} b_i \quad , \quad a_{ij} \in \mathbb{R}$$

different coefficients for different b_j, as each $\mathcal{A} b_j$ is a different vector in V

$$\Rightarrow \mathcal{A}v \underset{\circledast\circledast}{=} \sum_i \beta_i b_i \underset{\circledast}{=} \sum_j \alpha_j \mathcal{A} b_j \underset{\circledast\circledast\circledast}{=} \sum_j \alpha_j \sum_i a_{ij} b_i$$

$$= \sum_j \sum_i \alpha_j a_{ij} b_i = \boxed{\sum_{i,j} a_{ij} \alpha_j b_i}$$

$$\Rightarrow \beta_i = \sum_j a_{ij} \alpha_j$$ by uniqueness of representation

$$\beta = \begin{pmatrix} \beta_1 \\ | \\ \beta_n \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ | \\ \alpha_n \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & \text{———} & a_{1n} \\ | & & | \\ a_{n1} & \text{———} & a_{nn} \end{pmatrix}$$ is a <u>matrix</u>

We introduce the notation

$$\beta = A\alpha$$

matrix (-vector) multiplication

<u>Ex.:</u> $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\beta = A\alpha = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\beta = A\alpha = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 1\cdot 2 \\ 3\cdot 1 + 2\cdot 2 \end{pmatrix} \begin{array}{l} \text{1st row times } \alpha \\ \text{2nd row times } \alpha, \end{array}$$

with times as in scalar product.

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Example: V = vector space of polynomials of degree ≤ 2 with coefficients in $\mathbb{R}$

$$\mathcal{B} = \{1, x, x^2\} \quad \text{basis of } V$$

$$\mathcal{A} = \frac{d}{dx} \quad \text{differential operator}$$

To find corresponding matrix A, we need to look at how $\mathcal{A}$ affects the basis vectors $1, x, x^2$

We can then use $\mathcal{A} b_j = \sum_i a_{ij} b_i$ to construct A via the a_{ij}

$$\Rightarrow \quad \mathcal{A} 1 = 0 = 0 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \quad , \mathcal{A} x = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \quad , \mathcal{A} x^2 = 2x = 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2$$

So the matrix A has to have as entries the blue coefficients, with first row corresponding to $\mathcal{A}1$, second to $\mathcal{A}x$ and third to $\mathcal{A}x^2$.

$$\Rightarrow \quad A = \begin{array}{c} \begin{array}{ccc} & \mathcal{A}x & \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccc} \mathcal{A}1 & & \mathcal{A}x^2 \end{array} \end{array}$$

Now, take $p = 3x^2 - 2x + 7$

$$= 3 \cdot x^2 + (-2) \cdot x + 1 \cdot 7$$

so $\alpha = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ as coefficient vector of p for the basis $\mathcal{B}$.

$$\frac{dp}{dx} = 6x - 2 = 0 \cdot x^2 + 6 \cdot x + (-2) \cdot 1$$

so β as a result of $A\alpha$ is $\beta = \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix}$

We can check:

$$A\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \cdot 1 + \boxed{1 \cdot (-2)} + 0 \cdot 3 \\ 0 \cdot 1 + 0 \cdot (-2) + \boxed{2 \cdot 3} \\ 0 \cdot 1 + 0 \cdot (-2) + 0 \cdot 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 0 \end{pmatrix} \checkmark$$