Calculus and Elements of Linear Algebra I Session 23

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

6. Matrices

6.1 Introduction to matrices and link to linear operators

6.1.B: Basic matrix operations

Correspondence between operators and matrices

Operators $\mathcal{A}$ and $\mathcal{B}$	Matrices A and B
$(\mathcal{A}+\mathcal{B})v = \mathcal{A}v + \mathcal{B}v$	$\boxed{(A+B)_{ij} = a_{ij} + b_{ij}}$
$\mathcal{A}(\lambda v) = \lambda \mathcal{A} v$	$\boxed{(\lambda A)_{ij} = \lambda a_{ij}}$
	(with ij as indices)

$\Rightarrow$ $\mathcal{L}(V)$, the set of linear | $M(n \times n)$ is a vector operators on V, is a vector space

space ($n \times n$ matrices, n rows, n columns)

$$(\mathcal{A}\mathcal{B})v = \mathcal{A}(\mathcal{B}v)$$

AB where

$$\boxed{(AB)_{ij} = \sum_k a_{ik} b_{kj}}$$

makes sense for any $A \in M(n \times m)$, $B \in M(m \times p)$

$$AB \in M(n \times p)$$

(Matrix multiplication, more details later)

Transpose of matrix:

$$\boxed{(A^T)_{ij} = (a_{ji})}$$

switching columns and rows

e.g.: $\begin{pmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 3 & 1 \end{pmatrix}$

scalar product

row vector

column vector

In particular: $x \cdot y = x^T y = (x_1, \ldots, x_n) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$= x_1 y_1 + x_2 y_2 + \text{———} + x_n y_n$

For matrices with complex entries $(A \in \mathbb{C}^{n \times m})$:

Hermitian conjugate:

scalar product complex vectors

$\boxed{A^H = (A^*)^T}$, $x \bullet y = x^H y$

e.g.: $\begin{pmatrix} i & -i \\ 1 & 2 \end{pmatrix}^H = \begin{pmatrix} -i & i \\ 1 & 2 \end{pmatrix}^T = \begin{pmatrix} -i & 1 \\ i & 2 \end{pmatrix}$

<u>Fact:</u> $(AB)^T = B^T A^T$, $(AB)^H = B^H A^H$

(Proof: Try yourself)

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Matrix multiplication

$A = (a_{ij})$, $i = 1, \ldots, n$ n: # rows
$j = 1, \ldots, m$ m: # columns   of A

$B (b_{jk})$, $j = 1, \ldots, m$
$k = 1, \ldots, p$

$A \in M(n \times m)$, $B \in M(m \times p)$

m

Then: $$(AB)_{ik} = \sum_{j=1} a_{ij}\, b_{jk}$$

Ex.: ①

$$\underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}}_{B} = \begin{pmatrix} \overset{=2}{1\cdot 0 + 1\cdot 2} & \overset{=4}{1\cdot 1 + 1\cdot 3} \\ & \end{pmatrix}$$

for first row

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ \underset{=2}{-1\cdot 0 + 1\cdot 2} & \underset{=2}{-1\cdot 1 + 1\cdot 3} \end{pmatrix}$$

for second row

$$= \begin{pmatrix} 2 & 4 \\ 2 & 2 \end{pmatrix}$$

②

$$\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 5 \\ 1 & 0 & 2 & 6 \end{pmatrix} =$$

$$\begin{pmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 4 \\ 1 & 2 & 1 & 5 \\ 1 & 0 & 2 & 6 \end{pmatrix} =$$

$$= \begin{pmatrix} \overset{=6}{2\cdot1+1\cdot1+3\cdot1} & \overset{=4}{2\cdot1+1\cdot2+3\cdot0} & \overset{=5}{2\cdot(-1)+1\cdot1+3\cdot2} & \overset{=31}{2\cdot4+1\cdot5+3\cdot6} \\ \underset{=1}{-1\cdot1+1\cdot0+2\cdot1} & \underset{=-1}{-1\cdot1+0\cdot2+2\cdot0} & \underset{=5}{(-1)\cdot(-1)+0\cdot1+2\cdot2} & \underset{=8}{(-1)\cdot4+0\cdot5+2\cdot6} \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 4 & 5 & 31 \\ 1 & -1 & 5 & 8 \end{pmatrix}$$

<u>Note:</u> $\boxed{AB \neq BA}$ in general!

(BA might not even exist due to dimension, as in case above)