Calculus and Elements of Linear Algebra I — Session 24

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

6. Matrices

6.2 Solving systems of linear equations

Topic 6.2.A: Systems of linear equations

Take the system of linear equations

$$
\begin{aligned}
x_2 + 2x_3 - x_4 &= 1\\
x_1 + x_3 + x_4 &= 4\\
-x_1 + x_2 - x_4 &= 2\\
2x_2 + 3x_3 - x_4 &= 7
\end{aligned}
$$

This can be written in matrix form as

$$
Ax = \begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 0 \end{pmatrix} = b
$$

$$\underbrace{\qquad\qquad\qquad\qquad}_{=A} \quad \underbrace{\qquad}_{=x} \quad \underbrace{\qquad}_{=b}$$

So a system of linear equations is a matrix equation of the form $\boxed{Ax = b}$
(also called inhomogeneous equation for $b \neq 0$)
with x the vector of unknowns.

Suppose x and y are two solutions, i.e.

$$Ax = b \quad \text{and} \quad Ay = b$$

Take difference $\quad Ax - Ay = b - b$

$$\Rightarrow \quad A(x - y) = 0$$

So the difference vector $x - y = v$ between the solutions satisfies

$$Av = 0$$

(the homogeneous equation with right-hand side equal to zero)

Vice versa, if v solves $Av = 0$ and x solves $Ax = b$, then

$$\boxed{y := x + \lambda v} \text{ solves } Ay = A(x + \lambda v) = \overset{=b}{Ax} + \overset{=0}{\overset{=\lambda\cdot 0}{\lambda A v}}$$

$$= b$$

the inhomogeneous equation again.

Thus: The equation $Ax=b$

(i) has a unique solution iff $Av=0$ has only one solution $v=0$.

(ii) Otherwise: $Ax=b$ (a) may not have a solution at all

or

b) has many solutions that form a line, plane or hyperplane (i.e. n-dimensional plane)

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How to solve systems of linear equations?

We can perform elementary row operations on the full system:

(a) Re-order rows (i.e. change order of equations)

(b) Multiply rows with a non-zero scalar
(i.e. multiply equation by non-zero number)

(c) Add a multiple of one row to another
(adding / subtracting equations)

Example from before:

$$\overset{A}{\begin{pmatrix} 0 & 1 & 2 & -1 \\ 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 2 & 3 & -1 \end{pmatrix}} \overset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \overset{b}{\begin{pmatrix} 1 \\ 4 \\ 2 \\ 7 \end{pmatrix}}$$

Use augmented matrix $(A|b)$ of all coefficients a_{ij} and solutions b_i

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right) \quad \text{with the goal} \quad \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \tilde{b}_1 \\ 0 & 1 & 0 & 0 & \tilde{b}_2 \\ 0 & 0 & 1 & 0 & \tilde{b}_3 \\ 0 & 0 & 0 & 1 & \tilde{b}_4 \end{array}\right)$$

$$\underbrace{}_{\substack{A \\ \text{left side of} \\ \text{equations}}} \quad \underbrace{}_{\substack{b \\ \text{right side} \\ \text{of equations}}} \qquad\qquad \underbrace{}_{I} \quad \underbrace{}_{\tilde{b}}$$

with I the identity matrix, all entries zero except for ones along the diagonal.

The system $Ix = \tilde{b}$ corresponds to

$$\begin{aligned} x_1 \qquad\qquad &= \tilde{b}_1 \\ x_2 \qquad &= \tilde{b}_2 \\ x_3 &= \tilde{b}_3 \end{aligned}$$

so $x_1 = \tilde{b}_1$, $x_2 = \tilde{b}_2$, $x_3 = \tilde{b}_3$ and $x_4 = \tilde{b}_4$

$$x_4 = \tilde{b}_4$$

First step (for example; many options): reorder rows

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ -1 & 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right)$$

$$\xrightarrow[\text{row 2 + row 1 onto row 2}]{R2+R1 \Rightarrow R2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 1 & 2 & -1 & 1 \\ 0 & 2 & 3 & -1 & 7 \end{array}\right)$$

$$\xrightarrow[R4-2R2 \Rightarrow R4]{R3-R2 \Rightarrow R3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 1 & -1 & -5 \end{array}\right)$$

$$\xrightarrow[R_4-R_3 \Rightarrow R_4]{\substack{R_1-R_3 \Rightarrow R_1 \\ R_2-R_3 \Rightarrow R_2}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 9 \\ 0 & 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

does not contain additional information, i.e. $0=0$

This corresponds to:

$$\begin{aligned} x_1 \qquad\qquad &+ 2x_4 = 9 \\ x_2 \qquad &+ x_4 = 11 \\ x_3 &- x_4 = -5 \end{aligned}$$

x_1, x_2, x_3 are leading variables

and for x_4: $0 = 0$, i.e. free variable.

We have only 3 equations left that contain information but still have 4 unknowns.

So we can choose x_4 as we like, e.g. $x_4 = 0$

$\Rightarrow \quad w = \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix}$ is a solution to $Ax = b$.

However, it is not unique, as x_4 is free. For different choice of x_4, we get another solution.

So we have to find a vector v s.t. $Av = 0$ to find all solutions of the form $x = w + \lambda v$, $\lambda \in \mathbb{R}$

Last column of of the A part of $\circledast$ gives us that vector by setting x_4 to be a free parameter t:

$x_4 = t$

$$\Rightarrow \quad \begin{aligned} x_1 + 2t &= 0 \\ x_2 + t &= 0 \\ x_3 - t &= 0 \\ x_4 - t &= 0 \end{aligned} \qquad \text{setting } x_4 = t = -1 \quad \Rightarrow \quad \begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= -1 \\ x_4 &= -1 \end{aligned}$$

This corresponds to that last column, where zero entry is replaced by -1:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 9 \\ 0 & 1 & 0 & 1 & 11 \\ 0 & 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

replacing this by -1

This solves $Av = 0$, so we get in total:

$$x = \begin{pmatrix} 9 \\ 11 \\ -5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \\ -1 \end{pmatrix} \quad \text{a line in 4D space.}$$

This equation of the line is not unique, other solutions are possible but correspond to the same line.

The process of using row operations to find a solution to $Ax = b$ is called **Gaussian elimination**