Calculus and Elements of Linear Algebra I Session 25

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

6. Matrices

6.2 Solving systems of linear equations

Topic 6.2.B: Solution space of systems of linear equations

Taking, as an example, the following system

$$\begin{aligned} 2x_1 + 2x_2 &= 4 \\ 4x_1 + 4x_2 &= 8 \end{aligned} \Rightarrow \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 2 & 2 & 4 \\ 4 & 4 & 8 \end{array}\right) \xrightarrow[R1/2 \Rightarrow R1]{R2-2R1 \Rightarrow R2} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right)$$

focus on the diagonal entries and entries $\neq 0$

Def.: The first non-zero entry of a row is called a pivot provided there is no other pivot in that column.

Ideally, you want to do row operations until all pivots are on the diagonal and are 1.

In general, if not all pivots exist / not all columns have pivots, we expect that a solution only exists for some specific vectors b and the given matrix A (i.e. for some b there may <u>not</u> be a solution).
Sometimes reordering the rows may help to prevent pivots along diagonal to become zero. (more details, e.g., in *Numerical Methods*).

The matrix above, $\begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix}$, will have multiple solutions for some b, such has $b = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, but no solutions for some other, e.g. $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, as the augmented system will read

$$\left(\begin{array}{cc|c} 1 & 1 & \frac{1}{2} \\ 0 & 0 & -3 \end{array}\right)$$

with the last row a contradiction $0 \cdot x_1 + 0 \cdot x_2 = -3$

We call a system overdetermined, when we have more (linearly independent) equations than unknowns. Such systems do <u>not</u> have an exact solution. It is

<u>Eg.:</u>

$$\begin{pmatrix} 2 & 3 \\ 1 & -1 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

more rows than columns and rows are linearly independent, meaning one cannot construct (linearly) one from the other, e.g. $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 5 \end{pmatrix} \ \forall \lambda, \mu \in \mathbb{R}$

The solution to just the first two rows is

$$x_1 = \frac{7}{5}, \quad x_2 = -\frac{3}{5}$$

but this does not solve the last row

$$6 \cdot x_1 + 5 \cdot x_2 = \frac{42}{5} - \frac{15}{5}$$
$$= \frac{27}{5} \neq 3$$

We call a system underdetermined, when we have more unknowns than (linearly independent) equations. Such systems have multiple solutions. They are underconstrained.

<u>Eg.:</u>

$$\begin{pmatrix} 2 & 4 & 2 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

more columns than rows, i.e. more unknowns than equations. Rows are also linearly independent; if they were linearly dependent, there would be even fewer equations containing additional information.

The solutions are given by: $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\forall \lambda \in \mathbb{R}$

i.e. the solutions are on a line in 3D.

(see if you can find this as well)

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Def.: Let $A \in M(n \times m)$. Then

$$\operatorname{Ker} A = \{ v \in \mathbb{R}^m : Av = 0 \}$$

is kernel or null space or solution space of the homogeneous equation.
This is a vector space (check yourself)

$$\operatorname{Range} A = \{ Ax : x \in \mathbb{R}^n \}$$

rank A is given by the # of pivots ($\neq 0$) after Gaussian elimination / using row operations to eliminate all entries except those on the diagonal).

Let $v_1, \ldots, v_m \in \mathbb{R}^n$, $b \in \mathbb{R}^n$

We can then write

$$b = \underset{\text{unknown}}{\underbrace{x_1}} v_1 + \ldots + x_m \overset{\text{vector}}{v_m}$$

which is the same as writing

$$Ax = b \quad \text{with} \quad A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_m \\ | & & | \end{pmatrix}, \; A \in M(n \times m)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad \in \mathbb{R}^n \quad \in \mathbb{R}^n$$

We can make the following observations:

- Columns without a pivot ($\neq 0$) do not contribute to Range A (i.e. the respective entries of x for $Ax = b$ do not impact the set of choices for b)
  $\rightarrow$ to select a basis for the column space ($=$ Range A), use Gaussian elimination to find columns with pivots, then choose original columns of A as basis vectors.

- Columns without a pivot contribute to a basis of Ker A (i.e. the vector space of solutions to $Av = 0$).

- The vectors for the span of Ker A that come from, e.g., Gaussian elimination are linearly independent by construction and are therefore a basis of Ker A.

Rank-nullity theorem

Let $A \in M(n \times m)$, then

dim. of space of v for $Av=0$ dim. of space of possible b for $Ax=b$

$$\underbrace{\dim \operatorname{Ker} A}_{\text{nullity } A} + \underbrace{\dim \operatorname{Range} A}_{\text{rank } A} = m$$

number of columns of A

(where dim gives the number of basis vectors of the space)

rank

Columns with pivots contribute to dim Range A, columns without pivots contribute to dim Ker A.

nullity

Consequences: • $Ax=b$ has a solution for every $b \in \mathbb{R}^n$ iff $\dim \operatorname{Range} A = n$

- $Ax = b$ has a unique (i.e. exactly just one) solution if $b \in \text{Range } A$ and $\dim \text{Ker } A = 0$

- $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$ iff $n = m = \text{rank } A$ (i.e. a square matrix $\in M(n \times n)$ and all columns/rows are linearly independent)

In the last case we say A is non-singular or invertible and we write:

$\boxed{x = A^{-1} b}$, where A^{-1} is the inverse of A