Calculus and Elements of Linear Algebra I Session 26

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

6. Matrices

6.2 Solving systems of linear equations

Topic 6.2.C : Inverse of a matrix

We have $Ax = b$, $A \in M(n \times n)$, $b \in \mathbb{R}^n$

If A is invertible, we can do $x = A^{-1} b$
with A^{-1} the inverse of A.

- $\boxed{A A^{-1} = I}$ with I the identity matrix

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in M(n \times n)$$

- $\boxed{A^{-1} A = I}$
- A^{-1} is unique if it exists.

- $\boxed{\begin{array}{l} A^{-1} \text{ exists} \iff \underset{\text{(full rank)}}{\text{rank } A = n} \iff \text{every row (column) has a pivot} \\ \iff Ax = b \text{ has a unique solution for every } b \in \mathbb{R}^n \end{array}}$

(many more $\iff$, for example, A^{-1} exists $\iff$ all columns (rows) of A are linearly independent)

$\Rightarrow$ if $B \in M(n \times n)$ with the property $AB = I$, then $B = A^{-1}$

- $(A^{-1})^{-1} = A$; check: $\overbrace{(A^{-1})^{-1}}^{=B} A^{-1} = I$ because $(A^{-1})^{-1}$ is inverse of A^{-1}

  $\Rightarrow BA^{-1} = I \Rightarrow B = A$ by unique solvability and $AA^{-1} = I$

rules for transpose: $(AB)^T = B^T A^T$

- $(A^{-1})^T = (A^T)^{-1}$; check: $\underbrace{(A^{-1})^T}_{B} A^T = (A A^{-1})^T = I^T = I$

  $\Rightarrow B = (A^T)^{-1}$

- $(AB)^{-1} = B^{-1}A^{-1}$; check: $\underbrace{B^{-1}A^{-1}}_{C} \overbrace{A}^{=I} B = B^{-1} I B = B^{-1}B = I$

$\Rightarrow C = (AB)^{-1}$

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We are trying to solve $Ax=b$. Let A^{-1} be the inverse of $A \in M(n \times n)$.
Then:

$$x = A^{-1}b = A^{-1}(\underbrace{b_1}_{\in \mathbb{R}} \underbrace{e_1}_{\in \mathbb{R}^n} + b_2 e_2 + \ldots + b_n e_n),$$

where $e_i = (0, \ldots, 0, \underset{\substack{\uparrow \\ i\text{th position}}}{1}, 0, \ldots, 0)^T \in \mathbb{R}^n$

so $\quad x = b_1 A^{-1} e_1 + \ldots + b_n A^{-1} e_n$

From this we get the following procedure to find A^{-1}:

$$(A \mid I) \xrightarrow{\text{row operations}} (I \mid A^{-1})$$

Ex.:

$$A = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix}$$

$$\hookrightarrow \left(\begin{array}{ccc|ccc} 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[2R1 \to R2]{R2 \to R1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 2 & \frac{1}{2} & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R3 - 2R1 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow[\hookrightarrow -2R3 \to R3]{R3 - \frac{1}{2}R2 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right)$$

$$\xrightarrow[R1 - R3 \to R1]{R2 + R3 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 2 \\ 0 & 1 & 0 & 4 & 4 & -2 \\ 0 & 0 & 1 & 2 & 4 & -2 \end{array}\right)$$

$$\underbrace{}_{I} \qquad \underbrace{}_{A^{-1}}$$

We can check:

$$AA^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 1 \\ 2 & \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} -2 & -3 & 2 \\ 4 & 4 & -2 \\ 2 & 4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{2} - \frac{2}{2} & \frac{4}{2} - \frac{4}{2} & -\frac{2}{2} + \frac{2}{2} \\ -2+2 & -3+4 & 2-2 \\ -4+\frac{4}{2}+2 & -6+\frac{4}{2}+4 & 4-\frac{2}{2}-2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

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Application: Change of basis

Take the vector space $V = \text{span}\{\underset{e_1}{\underset{\parallel}{1}}, \underset{e_2}{\underset{\parallel}{x}}, \underset{e_3}{\underset{\parallel}{x^2}}\}$,

i.e. the vector space of polynomials of degree ≤ 2.

Take a second basis of this vector space

$b_1 = 2x - 1$, $b_2 = 2x + 1 + x^2$, $b_3 = x^2 - 1$

(you can show that any polynomial of degree ≤ 2 can be constructed by a linear combination of b_1, b_2, and b_3)

This new basis has coordinates with respect to e_1, e_2, e_3:

$$\underbrace{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}}_{=v_1}, \underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{=v_2}, \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{=v_3}$$

, e.g. for v_1: $-1 \cdot e_1 + 2 \cdot e_2$
$= -1 + 2x$
$= b_1$

A change of basis can be seen as a change of perspective.
A simplified example is whether one gives directions in metres or miles, or with respect to left and right or East and West.

If $p(x)$ has coordinates $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ w.r.t. e_1, e_2, e_3, what are its coordinates y w.r.t. b_1, b_2, b_3?

We have

$$x = y_1 b_1 + y_2 b_2 + y_3 b_3$$

or $\quad Sy = x \quad$ with $\quad S = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$

b_1 given as vector w.r.t. e_1, e_2, e_3

$\Rightarrow$ $\boxed{y = S^{-1} x}$

invertible, since columns form a basis! (i.e. are linearly independent

To compute S^{-1}:

$$\left(\begin{array}{ccc|ccc} -1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[2R1+R2 \to R2]{-R1 \to R1} \left(\begin{array}{ccc|ccc} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 4 & -2 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R3 \to R2]{R3+R1 \to R1} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & -2 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R3-4R2 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 4 & -2 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{R3-4R2 \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & -6 & 2 & 1 & -4 \end{array}\right)$$

$$\xrightarrow{\frac{R3}{-6} \to R3} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \end{array}\right.$$

$$\xrightarrow[R1-2R3 \to R1]{R2-R3 \to R2} \left(\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \,\middle|\, \underbrace{\begin{matrix} \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{matrix}}_{= S^{-1}} \right)$$