Calculus and Elements of Linear Algebra I — Session 3

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1. Functions

1.1 Numbers and Polynomials

Topic 1.1.C: Misc – Polynomial Long Division, Inequalities, Binomial Coefficients

Recall: Every polynomial can be factorized as $p(x) = a_n(x-z_n)(x-z_{n-1})\cdots(x-z_1)$, where $z_1,\dots,z_n$ are the n complex roots of the polynomial p of degree n.

Now suppose one of the roots is known. Then we can "divide out" this root and are left with a polynomial of degree $n-1$. This is called **polynomial long division.**

We discuss this method by example:

Consider $p(x) = 4x^3+3x^2-6x-1$. By inspection, we see that $z_1 = 1$ is a root $(4+3-6-1=0)$.
$\Rightarrow p(x) = (x-1)(4x^2+ax+b)$ Now we perform long division to find a and b:

We could also multiply out the right-hand side and compare coefficients, see exercise sessions.

$$
\begin{aligned}
&4x^3+3x^2-6x-1 = (x-1)\left(4x^2+7x+1\right).\\
-&\underline{(4x^3-4x^2)}\\
&\qquad 7x^2-6x\\
&\quad -\underline{(7x^2-7x)}\\
&\qquad\qquad x-1\\
&\qquad -\underline{(x-1)}\\
&\qquad\qquad\quad 0
\end{aligned}
$$

$\Rightarrow 0$ means finished

(Note: If we divide $p(x)$ by $(x-\alpha)$ and α is not a root, then a remainder polynomial will be left over.)

Finally, the roots of $4x^2+7x+1$ are $z_\pm = \frac{-7 \pm \sqrt{7^2-4\cdot 4\cdot 1}}{2\cdot 4} = -\frac{7}{8} \pm \frac{\sqrt{33}}{8}$.

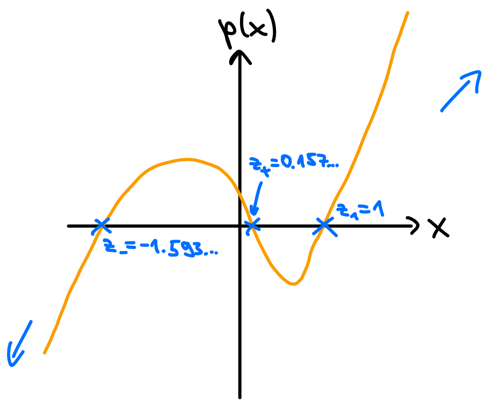
$\Rightarrow$ p has 3 real roots: $z_1 = 1$, $z_+ = -\frac{7}{8} + \frac{\sqrt{33}}{8}$, $z_- = -\frac{7}{8} - \frac{\sqrt{33}}{8}$.

Next: How to draw polynomials.

Previous example: $p(x) = 4x^3+3x^2-6x-1 = (x-1)\Big(x-\underbrace{\big(-\tfrac{7}{8}+\tfrac{\sqrt{33}}{8}\big)}_{z_+=0.157\ldots}\Big)\Big(x-\underbrace{\big(-\tfrac{7}{8}-\tfrac{\sqrt{33}}{8}\big)}_{z_-=-1.593\ldots}\Big)$.

We know:
- 3 points where $p(x)$ is 0.
- For very large x and very large negative x the term $4x^3$ dominates:
  - ↳ for x large positive, $4x^3$ is large positive,
  - ↳ for x large negative, $4x^3$ is large negative.

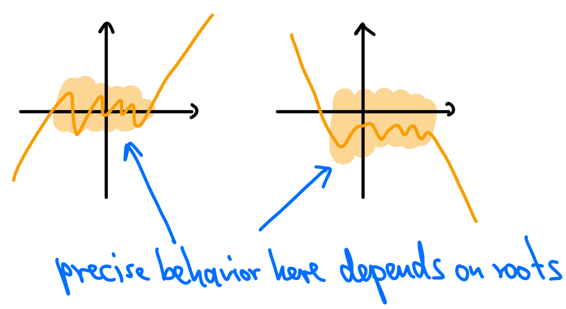
$\Rightarrow$



Note: From the behavior for very large positive/negative x we conclude:

$\boxed{\text{Any polynomial (with real coefficients) of odd degree has at least one real root.}}$

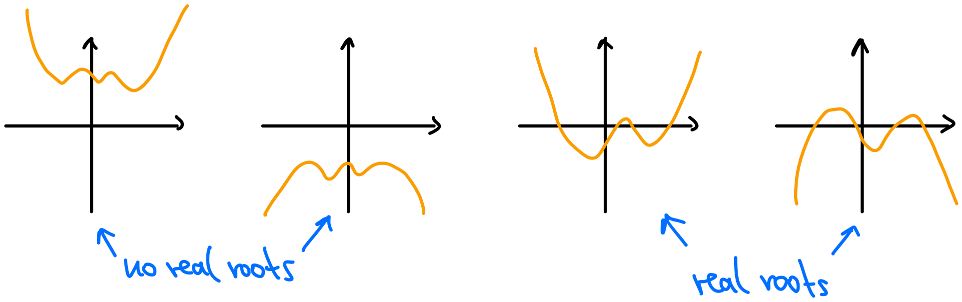
Why?



Note: Other reason: complex roots always come in pairs z, z^*, so at least one root needs to be real.

For a more thorough discussion we need limits and continuity $\longrightarrow$ will be discussed soon.

Note: Even polynomials ($n \geq 2$):



Next: Inequalities (studied by example).

• Linear. Example: For which x does $\underbrace{-3 \overset{\text{less than}}{<} \frac{1}{3}(7-2x) \overset{\text{less than or equal to}}{\leq} 4}_{\text{This means: } -3 < \frac{1}{3}(7-2x) \text{ and } \frac{1}{3}(7-2x) \leq 4.}$ hold?

We "solve" for x: $\quad -9 < 7-2x \leq 12$

$$\Rightarrow -16 < -2x \leq 5$$

$$\Rightarrow -8 < -x \leq \frac{5}{2}$$

if $-8 < -x$ then $x+8-8 < x+8-x$, i.e., $x < 8$!

$$\Rightarrow -\frac{5}{2} \leq x < 8 \qquad \left(8 > x \geq -\frac{5}{2}\right)$$

In interval notation: $x \in [-\frac{5}{2}, 8)$ (sometimes: $x \in [-\frac{5}{2}, 8[$).

$\frac{5}{2}$ included — 8 not included

• Quadratic. Example $x^2 - 4x - 12 > 0$?

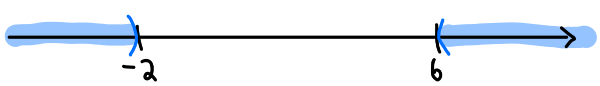
We compute the roots: $z_\pm = 2 \pm \sqrt{4+12} = \begin{cases} 6 \\ -2 \end{cases} \Rightarrow x^2 - 4x - 12 = (x+2)(x-6)$.

$\Rightarrow$ Need $\underbrace{x+2>0}_{x>-2}$ and $\underbrace{x-6>0}_{x>6}$, or $\underbrace{x+2<0}_{x<-2}$ and $\underbrace{x-6<0}_{x<6}$.

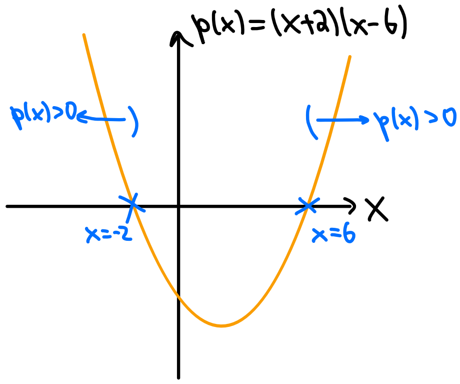
$\Rightarrow$ Solution: $x < -2$ or $x > 6$.

We can write this as $x \in \underbrace{\mathbb{R} \setminus [-2,6]}_{\mathbb{R} \text{ without the interval } [-2,6]} = (\overset{\text{symbol for "no lower bound"}}{-\infty}, -2) \cup (6, \underset{\text{symbol for "no upper bound"}}{\infty})$.

On the number line:



As a graph:



(For $x \in [-2,6]$ we have $p(x) \leq 0$.)

Finally: A brief review on binomial coefficients.

Goal: compute $(a+b)^n$

We have:

$$\begin{aligned}(a+b)^0 &= 1\\ (a+b)^1 &= a+b\\ (a+b)^2 &= a^2+2ab+b^2\\ (a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\ (a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\ &\vdots\end{aligned}$$

Note: the pattern

$$\begin{matrix} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \\ \vdots & & & & & & & & \vdots \end{matrix}$$

is called Pascal triangle.

The general formula is: $\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \dots \binom{n}{n}b^n}$.

This is called binomial formula (or binomial expansion).

The coefficients $\binom{n}{k}$ or ${}^nC_k$ ("n choose k") are called binomial coefficients.

One can show that $\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!}}$ $\quad (n! = n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1)$.