Calculus and Elements of Linear Algebra I — Session 4

Dr. Stephan Juricke, Prof. Sören Petrat (based on lecture notes by Marcel Oliver)
Jacobs University, Fall 2022

1. Functions

1.2 Functions and their Graphs

Topic 1.2.A: Equations, Functions and their Inverses, Graphs

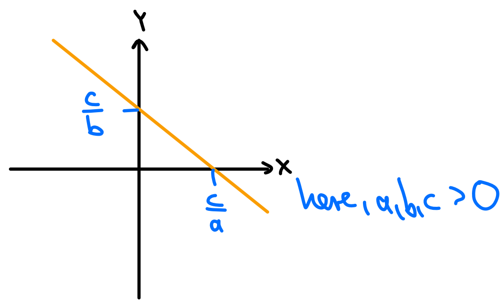
Equation of two variables x and y $:\Longleftrightarrow$ Arbitrary relationship between x and y.

Graph of an equation (of x and y) $:\Longleftrightarrow$ Set of all points (x,y) that satisfy the equation.

Examples:

- Line: $ax+by=c$ for given $a,b,c \in \mathbb{R}$.

Graph:

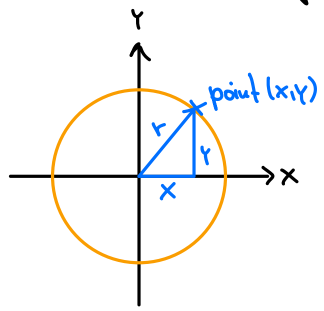


If $b \neq 0$, we can solve for y: $y = \frac{c}{b} - \frac{a}{b}x$ (here, we have expressed y as a function of x)

If $a \neq 0$, we can solve for x: $x = \frac{c}{a} - \frac{b}{a}y$ (here, we have expressed x as a function of y)

- Circle: $x^2+y^2=r^2$ for given $r>0$.

Graph:



Can we solve for y in terms of x here (or the other way around)?

Answer: Not in general, but, e.g., if we restrict to upper semi-circle, i.e., $y \geq 0$.

Then $y=\sqrt{r^2-x^2}$ for $x \in [-r,r]$.

Conclusion: Whenever we can solve for y in terms of x unambiguously, we can speak of y as a function of x. But we have to be careful which x and y are allowed!

So we define:

Definition:

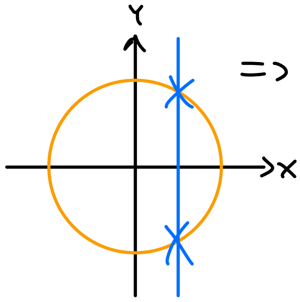
A function $f: A \to B$ is a rule that assigns to any $x \in A$ exactly one element $y \in B$.

We sometimes write $f: A \to B, x \mapsto f(x)$.

Furthermore:
- A is called **domain**.
- **Range(f)** $:= \{f(x) : x \in A\} \subset B$.
- The **graph of f** is the graph of the equation $y=f(x)$, $x \in A$.

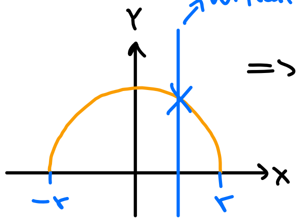
Note: $f(x)$ is unique, i.e., every vertical line through $(x,0)$ with $x \in A$ intersects the graph of f in exactly one point. (Sometimes called "vertical line test".)

Example:



$\Rightarrow$ the full circle is not the graph of one function, it fails the vertical line test

vertical line test ok

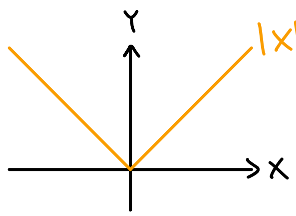


$\Rightarrow$ the upper semi-circle is the graph of

$$f: \underbrace{[-r,r]}_{\text{domain}} \longrightarrow \underbrace{[0,r]}_{B=\text{range}(f)\text{ here}},\ x \longmapsto f(x) = \sqrt{r^2 - x^2}$$

Next: Standard functions we need to know.

- $\boxed{\text{Absolute value.}}$

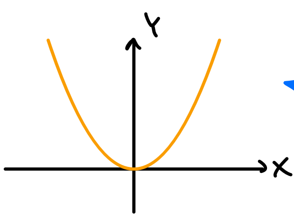


$$\text{abs}: \mathbb{R} \longrightarrow [0,\infty),\ x \longmapsto |x| = \sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Note: for $z = x + iy \in \mathbb{C}$ we define $|z| := \sqrt{z^* z} = \sqrt{x^2 + y^2}$.

- $\boxed{\text{Parabola.}}$

$f(x) = x^2$.

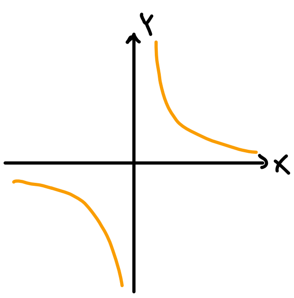


note: cannot solve for x in terms of y here ("horizontal line test" fails)

Here, domain$(f) = \mathbb{R}$, range$(f) = [0,\infty)$

- $\boxed{\text{Hyperbola.}}$

$f(x) = \frac{1}{x}$.

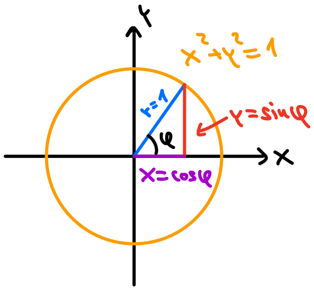


Here, domain$(f) = \mathbb{R}\setminus\{0\}$, range$(f) = \mathbb{R}\setminus\{0\}$.

Here, we can also express x in terms of y: $x = \frac{1}{y}$.

Trigonometric functions sin, cos, tan.

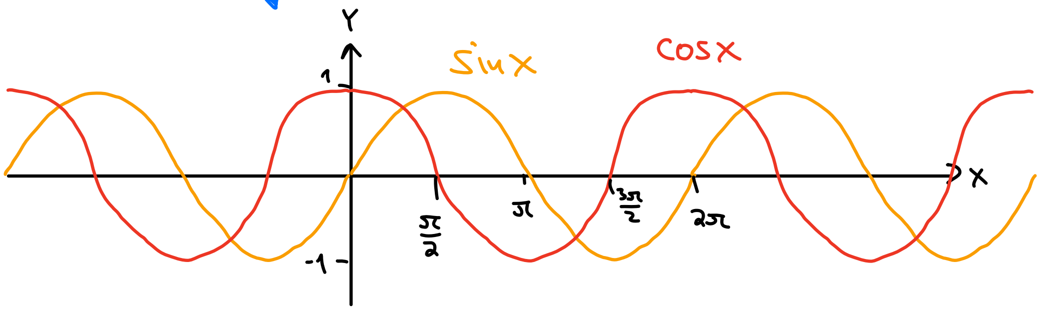
Unit circle:



(Note: $\sin^2\varphi+\cos^2\varphi=1$ ✓)

Recall: $90°$ angle $\hat{=}\ \frac{\pi}{2}$, $180° \hat{=}\ \pi$, $270° = \frac{3\pi}{2}$, $360° = 2\pi$.

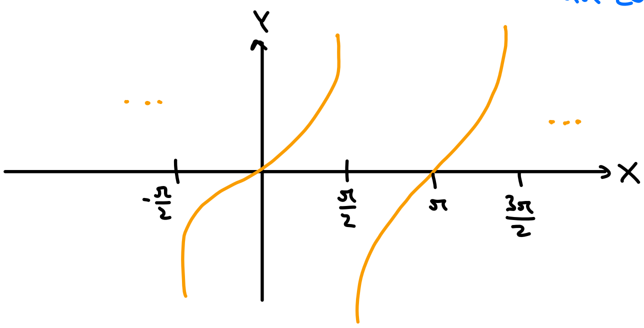
Graphs of $\sin x$, $\cos x$:



$\mathrm{Domain}(\sin)=\mathrm{domain}(\cos)=\mathbb{R}$, $\mathrm{range}(\sin)=\mathrm{range}(\cos)=[-1,1]$.

Furthermore: $\tan\varphi=\frac{\sin\varphi}{\cos\varphi}$, with $\mathrm{domain}(\tan)=\mathbb{R}\setminus\underbrace{\left\{\frac{\pi}{2}+n\pi : n\in\mathbb{Z}\right\}}_{\text{all zeroes of cos need to be excluded}}$.

Graph of tan:



$\Rightarrow \mathrm{range}(\tan)=\mathbb{R}$.

Exponential function.

For $n\in\mathbb{N}$, $\mathbb{R}\ni a>0$ we have $a^n := \underbrace{a\cdot a\cdots a}_{n \text{ times}}$.

It satisfies $a^{n+m}=a^n a^m$. (*)

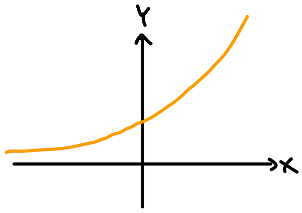
Similar to before we can now define a^x for $x\in\mathbb{Z}$, $x\in\mathbb{Q}$, $x\in\mathbb{R}$ while keeping (*) true.

We will do this in the exercises.

Result: a^x can be defined for $a>0, x\in\mathbb{R}$, and satisfies $a^{x+y}=a^x a^y$.

Note: We also have $(a^x)^y = a^{xy} = (a^y)^x$ $\quad (\neq a^{(x^y)})$

Graph of a^x:



$\Rightarrow$ domain$(a^x)=\mathbb{R}$, range$(a^x)=(0,\infty)$.

Next: Inverse of a function.

Definition:

If $f: A\to B$, then $g: B\to A$ is called the $\boxed{\text{inverse of } f}$ if

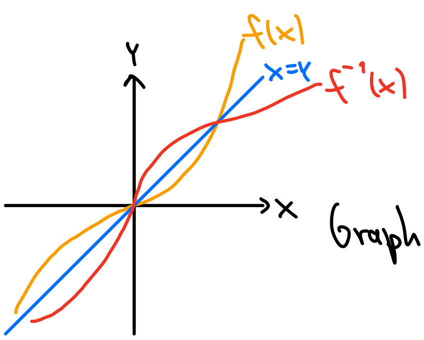
- $g(f(x)) = x \quad \forall x\in A,\quad (*)$
- $f(g(y)) = y \quad \forall y\in B.\quad (**)$

Note: If range $f = B$ and $(*)$ holds, then $(**)$ holds automatically.

(Why: If $y\in B$, then $\exists x\in A$ s.t. $f(x)=y$ (works bc. range $f=B$). Then $f(g(y)) = f(\underbrace{g(f(x))}_{=x \text{ by } (*)}) = f(x) = y$.)

Note: g is usually denoted by f^{-1}. $\leftarrow$ this does not mean $\frac{1}{f}$ here!

Graphically:



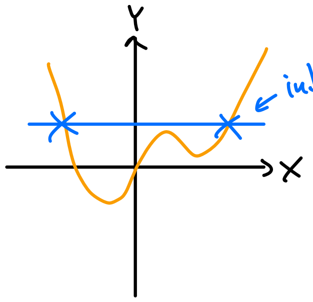
Graph of $y=f^{-1}(x) \Leftrightarrow \underbrace{f(y)=x}$.

Same as $f(x)=y$ but with x and y interchanged.

$\Rightarrow$ "Graph of $y=f^{-1}(x)$" = "Graph of $y=f(x)$ reflected at $y=x$ (blue line)".

Careful: If f fails the horizontal line test, then f is not invertible.

E.g.:



intersects at more than one point $\Rightarrow$ inverse not well defined.

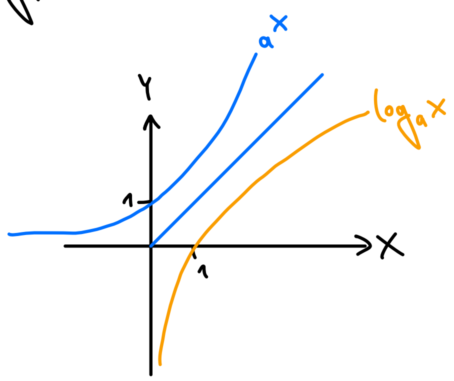
One more important function:

- $\boxed{\text{Logarithm.}}$

For $f(x) = a^x$, $a > 0$, $x \in \mathbb{R}$ we call the inverse the logarithm to base a: $\log_a y$.

$\Rightarrow \log_a a^x = x$.

Graph:



Note: $\log_a 1 = 0$ (bc. $a^0 = 1$).

Note: Since $a^{\hat{x}+\hat{y}} = a^{\hat{x}} a^{\hat{y}}$, we have $\underbrace{\log_a a^{\hat{x}+\hat{y}}}_{= \hat{x}+\hat{y}} = \log_a(a^{\hat{x}} a^{\hat{y}})$.

Thus, if we call $a^{\hat{x}} = x$ ($\hat{x} = \log_a x$), $a^{\hat{y}} = y$ ($\log_a \hat{y} = y$), then $\boxed{\log_a(xy) = \log_a x + \log_a y.}$