Calculus and Elements of Linear Algebra I Session 5

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# 1. Functions

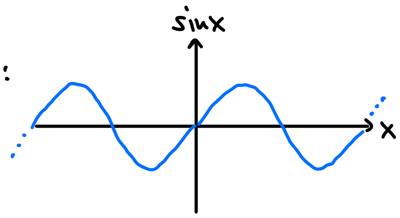
## 1.3 Limits and Continuity

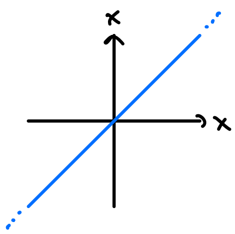
### Topic 1.3.A: Definition of Limits and Limit Laws

Motivation:

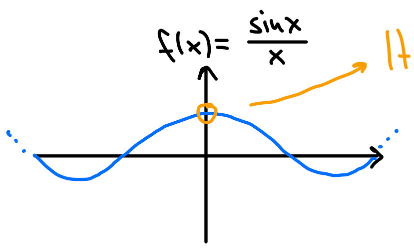
- $f(x) = \frac{\sin x}{x}$ has domain $\mathbb{R} \setminus \{0\}$. (since we can't divide by zero)

Plots:



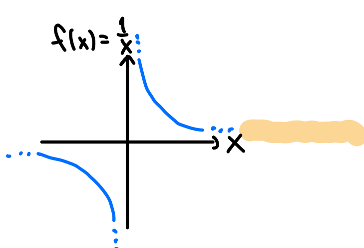




A plot of $\frac{\sin x}{x}$ reveals:




It appears $f(x)$ gets arbitrarily close to 1 as $x$ gets close to zero.

- $f(x) = \frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$:




It appears $f(x)$ tends to zero as $x$ gets larger and larger

Next: Make such statements precise.

Informal definition:

We call $L\in\mathbb{R}$ the limit of $f(x)$ as $x$ approaches $x_0$ if

$f(x)$ can be made arbitrarily close to $L$ for $x$ sufficiently close to $x_0$.

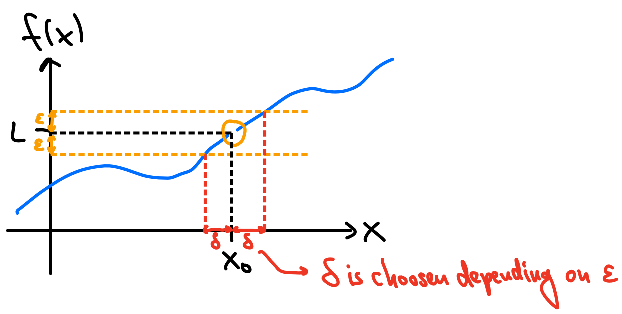
We write: $\boxed{\lim\limits_{x\to x_0} f(x) = L \quad \text{or} \quad f(x) \xrightarrow[x\to x_0]{} L.}$

More formally:

**Definition:**

Let $x_0\in(a,b)$ and $f:(a,b)\setminus\{x_0\}\to\mathbb{R}$. Then $\lim\limits_{x\to x_0} f(x) = L$ if:

$\underbrace{\forall}_{\text{for all}}\varepsilon>0$ $\underbrace{\exists}_{\text{there exists}}\delta>0$ such that $\underbrace{\forall x\in(a,b)\setminus\{x_0\}}_{\text{to ensure } f(x) \text{ makes sense } (x \text{ is in domain of } f)}$ with $0<|x-x_0|<\delta$ we have $|f(x)-L|<\varepsilon$.




Simple example: For $f(x)=x$, then $f(x)\xrightarrow[x\to x_0]{} L = x_0$, or $\lim\limits_{x\to x_0} x = x_0$.

Here, given $\varepsilon$, we can choose $\delta=\varepsilon$ because then: if $|x-x_0|<\delta=\varepsilon \Rightarrow |\underbrace{f(x)}_{=x}-\underbrace{L}_{=x_0}|<\varepsilon$.

More interesting example: $f(x)=\frac{1}{x}$, for $x\to x_0$ with $x_0\neq 0$.

How to choose $\delta$ for given $\varepsilon$? See exercise sessions.

Next: The limit laws.

**Proposition:**

(i) $\lim\limits_{x \to x_0} \left( f(x) + g(x) \right) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x)$,

(ii) $\lim\limits_{x \to x_0} f(x) g(x) = \lim\limits_{x \to x_0} f(x) \lim\limits_{x \to x_0} g(x)$,

(iii) $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to x_0} f(x)}{\lim\limits_{x \to x_0} g(x)}$ if $\lim\limits_{x \to x_0} g(x) \neq 0$.

These laws can be easily proved from the definition above.

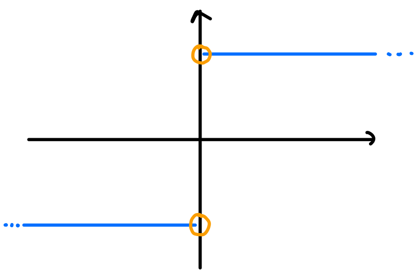
Example: $\lim\limits_{s \to 2} \dfrac{s^2 - 4}{s^2 + s - 6} = \lim\limits_{s \to 2} \dfrac{(s-2)(s+2)}{(s-2)(s+3)} = \lim\limits_{s \to 2} \dfrac{s+2}{s+3} = \dfrac{\lim\limits_{s \to 2}(s+2)}{\lim\limits_{s \to 2}(s+3)} = \dfrac{4}{5}$.

Can't apply (III) directly bc. denominator becomes zero

(iii)

Next: left and right limits.

Consider the following example: $f(x) = \dfrac{|x|}{x}$ with domain $\mathbb{R} \setminus \{0\}$:



Here, $\lim\limits_{x \to 0} f(x)$ does not exist, because for every $x > 0$ we have

$|f(x) - f(-x)| = |1 - (-1)| = 2$, i.e., it is not possible to find $L$ s.t. $|f(x) - L| < \varepsilon \ \forall \varepsilon > 0$, no matter how close $x$ is chosen to zero.

But it makes sense to define limits from the left or right:

$\lim\limits_{\substack{x \to 0 \\ \boxed{x > 0}}} f(x) = 1$ and $\lim\limits_{\substack{x \to 0 \\ \boxed{x < 0}}} f(x) = -1$ here.

↑ use only $x > 0$ in the $\varepsilon$-$\delta$-definition above

Definition:

(i) The limit from the right is $\lim\limits_{\substack{x \to x_0 \\ x > x_0}} f(x) \equiv \underbrace{\lim_{x \downarrow x_0}}_{\text{"limit from above"}} f(x) \equiv \lim\limits_{x \to x_0^+} f(x)$.

(ii) The limit from the left is $\lim\limits_{\substack{x \to x_0 \\ x < x_0}} f(x) \equiv \underbrace{\lim_{x \uparrow x_0}}_{\text{"limit from below"}} f(x) \equiv \lim\limits_{x \to x_0^-} f(x)$.

Note: lim exists if and only if both left- and right-sided limits exist and coincide.