Calculus and Elements of Linear Algebra I — Session 6

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1. Functions

1.3 Limits and Continuity

Topic 1.3.B: Asymptotes and Limits of the Exponential Function

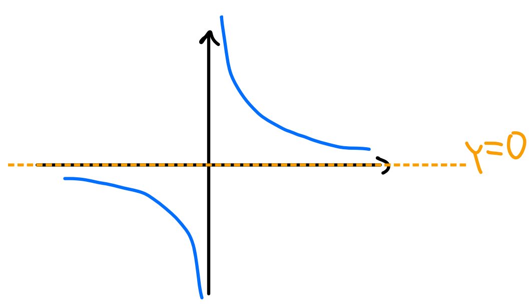
Asymptotes can capture the behavior of functions for very large x, e.g., $\frac{1}{x}$ tends to zero for larger and larger x.

Definition:

We write
- $\underbrace{\lim_{x\to\infty}}_{\text{"x to infinity"}} f(x) = L$ if $\lim_{y\searrow 0} f\left(\frac{1}{y}\right) = L$,
- $\lim_{x\to-\infty} f(x) = L$ if $\lim_{y\nearrow 0} f\left(\frac{1}{y}\right) = L$,

and call the line $y = L$ a **horizontal asymptote**.

Example: $\lim_{x\to\infty} \frac{1}{x} = \lim_{y\searrow 0} \frac{1}{1/y} = \lim_{y\searrow 0} y = 0$, i.e., $y=0$ is the horizontal asymptote for $\frac{1}{x}$ as $x\to\infty$.



(Also $\lim_{x\to-\infty} \frac{1}{x} = 0$.)

More examples in exercise sessions.

Caution: One cannot just "subsitute $x=\infty$" into an expression. The following three limits are all different:

e.g., $\frac{\infty}{\infty}=1$ is not a valid expression

- $\lim\limits_{x\to\infty} \frac{x}{x} = \lim\limits_{x\to\infty} 1 = 1$.
- $\lim\limits_{x\to\infty} \frac{x}{x^2} = \lim\limits_{x\to\infty} \frac{1}{x} = 0$.
- $\lim\limits_{x\to\infty} \frac{x^2}{x} = \lim\limits_{x\to\infty} x$ does not exist because x grows without bounds.

Next: Capture behavior such as $\lim\limits_{x\searrow 0} \frac{1}{x}$.

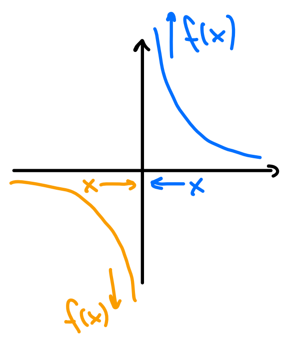
Definition:

If $f(x)$ increases (decreases) without bounds as $x \nearrow x_0$ $(x\searrow x_0)$, we write $\underbrace{\lim\limits_{x\nearrow x_0 \; (x\searrow x_0)} f(x) = \infty \; (-\infty)}$ and say that f has a $\boxed{\text{vertical asymptote}}$ $x=x_0$.

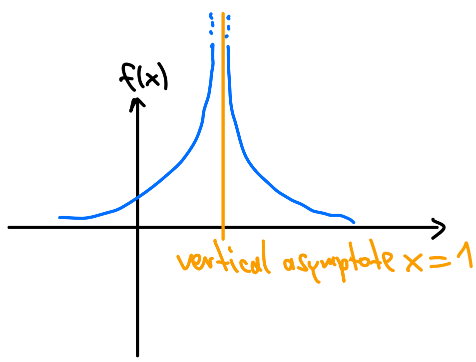
Still, we say that here the limit does not exist. We cannot compute, i.e., do algebraic manipulations with ∞.

Examples:

- $\lim\limits_{x\searrow 0} \frac{1}{x} = \infty$, $\lim\limits_{x\nearrow 0} \frac{1}{x} = -\infty$.



- $\lim\limits_{x\searrow 1} \frac{1}{(x-1)^2} = \infty$, $\lim\limits_{x\nearrow 1} \frac{1}{(x-1)^2} = \infty$



$\Rightarrow$ Here we would thus write $\lim\limits_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

Next: Limits involving the exponential function.

Question: What is $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x}$? ← Both x and e^x tend to ∞.

We claim that $\frac{n}{e^n} \leq \left(\frac{2}{e}\right)^n \quad \forall n \in \mathbb{N}$ holds. (*)

This would imply $0 \leq \lim_{n\to\infty} \frac{n}{e^n} \leq \lim_{n\to\infty} \left(\frac{2}{e}\right)^n = 0$, i.e., $\lim_{n\to\infty} \frac{n}{e^n} = 0$ (by the squeeze law).

since $e = 2.718\ldots$, i.e. $\frac{2}{e} < 1$

For $x \in \mathbb{R}$ instead of $n \in \mathbb{N}$ one can easily modify the argument (rounding to nearest integer).

We still need to prove (*). We prove this by the method of induction:

Proof Method: Induction

We aim at proving a claim for all natural numbers $n \geq n_0 \in \mathbb{N}_0$.

Step 1: Show that the claim holds for $n = n_0$. (Usually easy, since usually $n_0 = 0$ or 1.)

Step 2: We assume the claim holds for some $n \in \mathbb{N}$ and then prove that it holds for $n+1$.

By step 1 the claim holds for n_0, by step 2 it holds for n_0+1, by step 2 again it holds for n_0+2, and so on $\Rightarrow$ Claim holds for all $n \geq n_0$.

Our example: Claim: $\frac{n}{e^n} \leq \left(\frac{2}{e}\right)^n$

Step 1: $n=1$: $\frac{1}{e} \leq \frac{2}{e}$ holds ✓

Step 2: Assume $\frac{n}{e^n} \leq \left(\frac{2}{e}\right)^n$ holds. (*)

Then $$\frac{n+1}{e^{n+1}} = \frac{n}{e^{n+1}} + \frac{1}{e^{n+1}} \underset{1\leq n}{\leq} \frac{2n}{e^{n+1}} = \frac{2}{e}\underbrace{\left(\frac{n}{e^n}\right)}_{\leq \left(\frac{2}{e}\right)^n \text{ by } (*)} \leq \frac{2}{e}\left(\frac{2}{e}\right)^n = \frac{2^{n+1}}{e^{n+1}} \quad ✓$$

$\Rightarrow$ Claim holds.

Note: From $\frac{x}{e^x} \xrightarrow[x\to\infty]{} 0$ one can easily show the following more general and very important/useful statement:

$$\boxed{\lim_{x\to\infty} \frac{x^\alpha}{e^x} = 0 \text{ for every } \alpha>0 \text{, i.e., } e^x \text{ grows faster than any power of } x,}$$

The inverse result is:

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x^\alpha} = 0 \text{ for every } \alpha>0 \text{, i.e., } \ln x \text{ grows slower than any power of } x.}$$

Note: The latter can be written as $\lim_{x\to 0} x^\alpha \ln x = 0 \ \forall \alpha>0.$