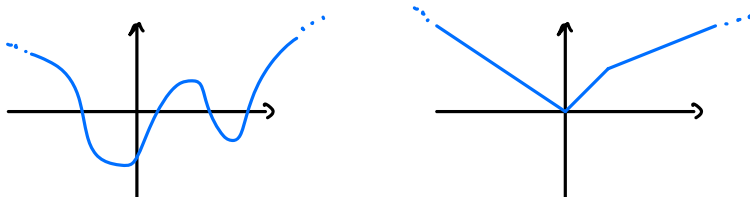


1. Functions1.3 Limits and Continuity

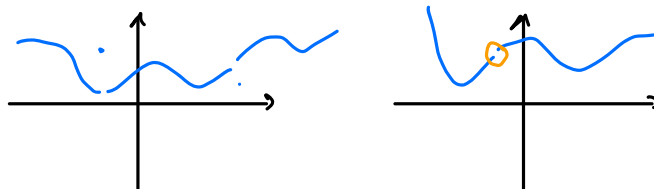
## Topic 1.3.C: Continuity and the Intermediate Value Theorem

Informally: "A function is continuous if it can be drawn without lifting the pen."

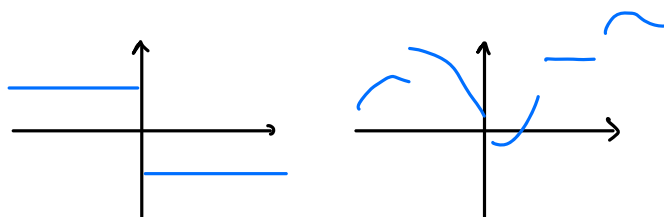
E.g., • continuous functions:



• functions with removable discontinuities:



• functions with non-removable discontinuities:



Formal definition with limits:

Definition:

A function  $f: (a, b) \rightarrow \mathbb{R}$  is **continuous** at  $x_0 \in (a, b)$  if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$ .

If  $f$  is continuous at every point in its domain, we say " $f$  is continuous".

Discontinuous functions can have:

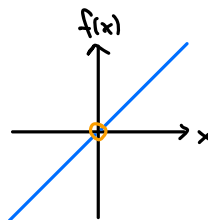
a) Removable discontinuities:  $f(x_0) \neq L$  or  $f(x_0)$  not defined, but  $\lim_{x \rightarrow x_0} f(x) = L$ .

Then we can remove the discontinuity by defining  $f(x_0) = L$ .

b) Non-removable discontinuities such as jumps.

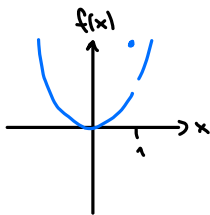
Simple examples:

•  $f(x) = \frac{x^2}{x}$  with domain  $\mathbb{R} \setminus \{0\}$ .



We can remove the discontinuity by defining  $f(0) := \lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$ .

•  $f(x) = \begin{cases} x^2 & \text{for } x \neq 1, \\ 2 & \text{for } x = 1. \end{cases}$



Discontinuity at  $x=1$  can be removed by setting  $f(1) := \lim_{x \rightarrow 1} x^2 = 1$ .

Note: Without proof, we can use that the following standard functions are continuous:

• polynomials  $\sum_{k=0}^n a_k x^k$ ,

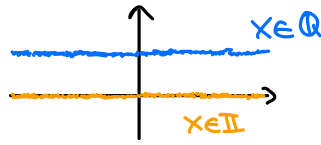
• trigonometric functions  $\sin x, \cos x, \tan x$  with domain  $(-\frac{\pi}{2}, \frac{\pi}{2})$ ,

•  $\exp x = e^x$ ,  $\ln x$  with domain  $(0, \infty)$

• hyperbola  $\frac{1}{x}$  with domain  $\mathbb{R} \setminus \{0\}$ . ( $\frac{1}{x}$  is continuous at all  $x \in \mathbb{R} \setminus \{0\}$ , but it is discontinuous at  $x=0$ , where it is not well-defined (vertical asymptote).)

Also note: There are functions on  $\mathbb{R}$  that are discontinuous everywhere, e.g.,

$$f(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q}, \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q} = \mathbb{I}. \end{cases}$$



Once we know that a function is continuous, we can use lots of useful theorems. (We state them here without proof, or with "proof by picture".)

Theorem:

If  $f: \mathbb{R} \rightarrow \mathbb{R}$  is continuous, and  $g: (a,b) \rightarrow \mathbb{R}$  has  $\lim_{x \rightarrow x_0} g(x) = L$  for  $x_0 \in (a,b)$ , then  $\lim_{x \rightarrow x_0} f(g(x)) = f(L) = f(\lim_{x \rightarrow x_0} g(x))$ .

↳ i.e., we can move limits inside of continuous functions.

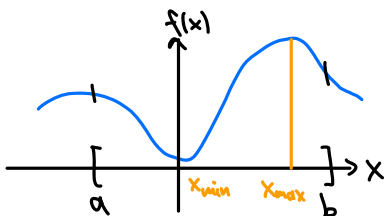
E.g.,  $\lim_{x \rightarrow 3} \sin\left(\frac{x^2-9}{x-3}\right) = \sin\left(\lim_{x \rightarrow 3} \frac{x^2-9}{x-3}\right) = \sin(6)$ .

$$= \frac{(x-3)(x+3)}{x-3} = x+3$$

Extreme Value Theorem:

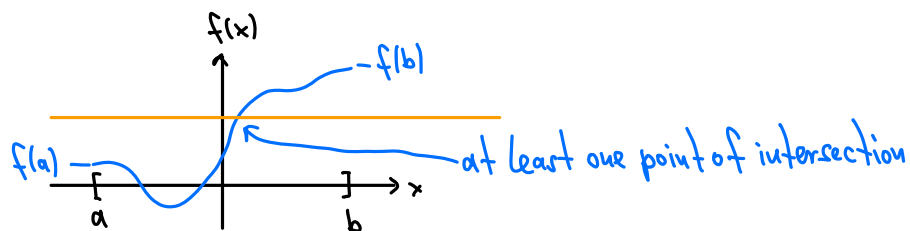
If  $f: [a,b] \rightarrow \mathbb{R}$  is continuous, then  $f$  assumes its minimum and maximum, i.e., there exist closed interval is important here

- $x_{\min} \in [a,b]$  s.t.  $f(x) \geq f(x_{\min}) \forall x \in [a,b]$ ,
- $x_{\max} \in [a,b]$  s.t.  $f(x) \leq f(x_{\max}) \forall x \in [a,b]$ .



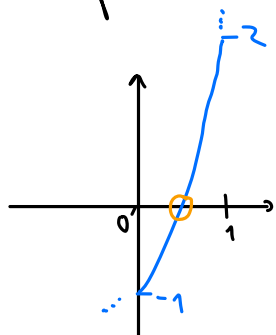
## Intermediate Value Theorem:

If  $f: [a, b] \rightarrow \mathbb{R}$  is continuous, then it assumes all intermediate values between  $f(a)$  and  $f(b)$ , i.e., for every  $y \in [f(a), f(b)]$  (or  $[f(b), f(a)]$  if  $f(b) < f(a)$ ) there is  $x \in [a, b]$  s.t.  $y = f(x)$ .



Example:  $p(x) = x^3 + x^2 + x - 1$

We have  $p(1) = 1 + 1 + 1 - 1 = 2$ ,  $p(0) = -1$ , so we know  $p$  has at least one root in  $[0, 1]$ .



This leads to the bisection method, see example section.

From the limit laws, we immediately get:

## Proposition:

If  $f$  and  $g$  are continuous at  $x_0$ , then also  $f+g$ ,  $f \cdot g$ , and  $\frac{f}{g}$  for  $g(x_0) \neq 0$ , are continuous at  $x_0$ .

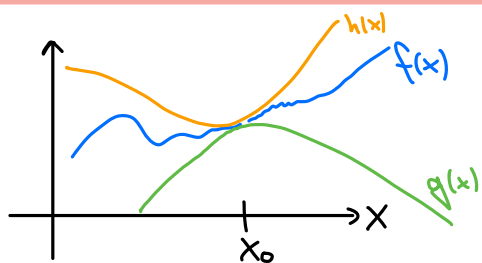
Also,  $f(g(x)) = (f \circ g)(x)$  is continuous at  $x_0$  if  $g$  is cont. at  $x_0$  and  $f$  is cont. at  $g(x_0)$ .

Furthermore, the squeeze law immediately implies:

Theorem:

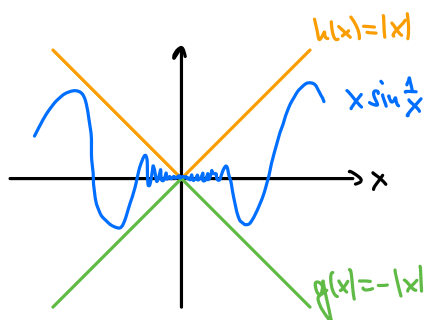
Let  $x_0 \in (a, b)$ , and  $f, g, h: (a, b) \setminus \{x_0\} \rightarrow \mathbb{R}$  be such that  $g(x) \leq f(x) \leq h(x)$  and

$\lim_{x \rightarrow x_0} g(x) = L \leq \lim_{x \rightarrow x_0} h(x)$ . Then  $\lim_{x \rightarrow x_0} f(x) = L$ .



Example:  $f(x) = x \sin \frac{1}{x}$  with domain  $\mathbb{R} \setminus \{0\}$ :

Note:  $|f(x)| = |x| \underbrace{|\sin \frac{1}{x}|}_{\leq 1} \leq |x|$



$\Rightarrow \lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$  and  $f$  is continuous on  $\mathbb{R}$  with the definition  $f(0) = 0$ .