Calculus and Elements of Linear Algebra I Session 8

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

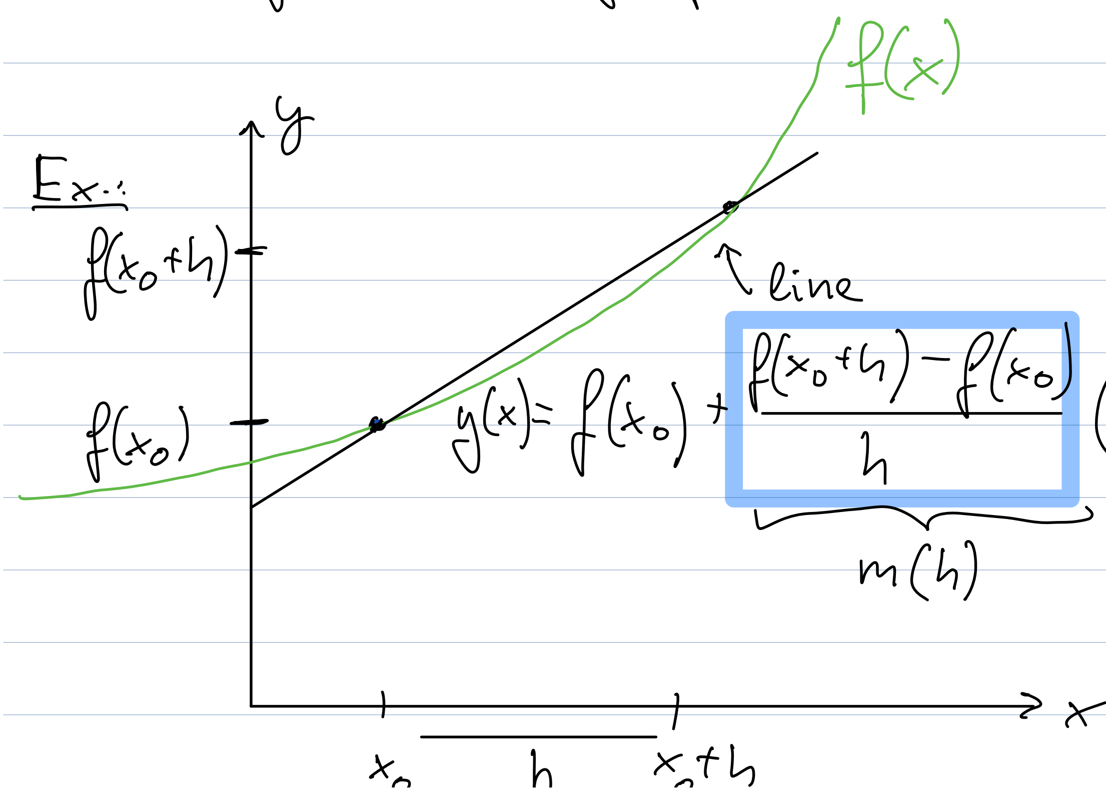
2. Derivatives

2.1 Introduction to derivatives and their properties

Topic 2.1.A: General definition

Derivative gives the slope of a function at a certain point.

Ex.:



$$y(x) = f(x_0) + \underbrace{\boxed{\frac{f(x_0+h)-f(x_0)}{h}}}_{m(h)}(x-x_0)$$

If h is small, the line y can be seen as a **linear approximation** to $f(x)$ at x_0.

<u>Idea:</u> Let $h \to 0$, expect that $m(h)$ converges, and the limit can be interpreted as the **slope of f** at point x_0.

(slope of tangent to f at x_0)

<u>Def.:</u> The derivative of $f:(a,b) \to \mathbb{R}$ at a point $x \in (a,b)$ is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

<u>Ex.:</u> $f(x) = \sin(x)$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$ (using definition)

and use: $\boxed{\sin(x+h) = \sin x \cdot \cosh + \cos x \cdot \sin h}$ angle addition formula

$$\Rightarrow f'(x) = \lim_{h \to 0} \left(\sin x \underbrace{\frac{\cos h - 1}{h}} + \cos x \underbrace{\frac{\sin h}{h}} \right)$$

$$\xrightarrow[h\to 0]{(*)} 0 \qquad \xrightarrow[h\to 0]{} \cos x$$

because $\boxed{\lim_{h\to 0} \frac{\sin h}{h} = 1}$

<u>Now, proof of (*):</u>

Take $g(h) = \frac{1-\cos h}{h^2}$ and use $\boxed{1-\cos h = 2\sin^2 \frac{h}{2}}$

$$= 2\,\frac{\sin^2 \frac{h}{2}}{h^2} \underset{h=2y}{\overset{\text{set } y=\frac{h}{2}}{=}} 2\,\frac{\sin^2 y}{(2y)^2} = \frac{1}{2}\underbrace{\left(\frac{\sin y}{y}\right)^2}_{\to 1 \text{ as } y\to 0 \text{ or } h\to 0}$$

$$\xrightarrow[h\to 0]{} \frac{1}{2}$$

If $\lim_{h\to 0} \frac{1-\cos h}{h^2} = \frac{1}{2}$ then $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$

because $\lim_{h\to 0} \frac{1-\cos h}{h^2} \cdot h = \underbrace{\lim_{h\to 0}\left(\frac{1-\cos h}{h^2}\right)}_{=\frac{1}{2}} \cdot \underbrace{\lim_{h\to 0}(h)}_{=0} = \frac{1}{2}\cdot 0 = 0$

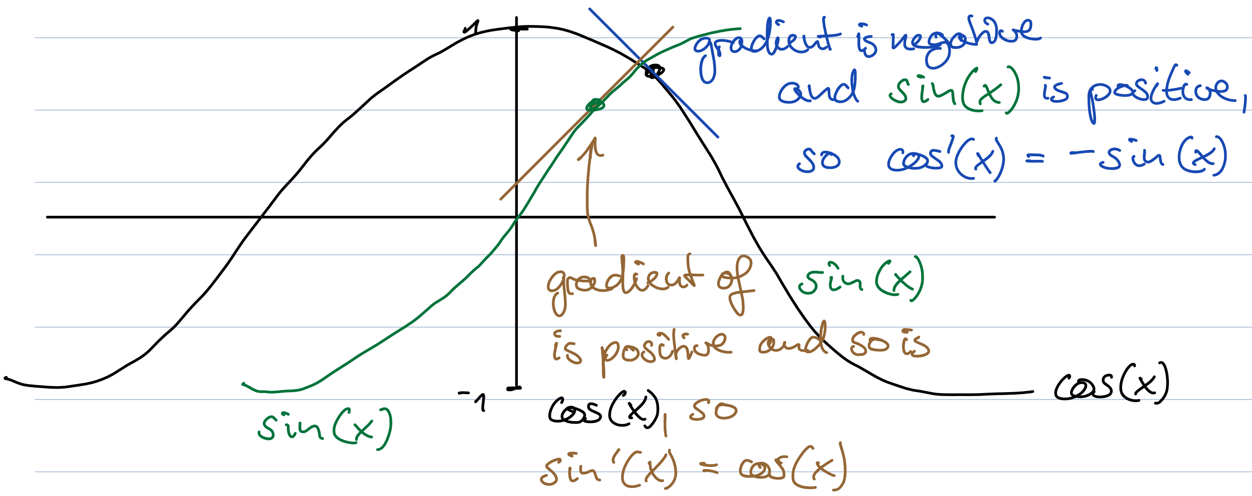
and so $\boxed{\lim_{h\to 0} \frac{\cos h - 1}{h} = 0}$

<u>Conclusion:</u> $f'(x) = \lim_{h\to 0}\left(\underbrace{\sin x\,\frac{\cos h - 1}{h}}_{\to 0} + \underbrace{\cos x\,\frac{\sin h}{h}}_{\to \cos x}\right)$

$$= \cos x$$

$$\Rightarrow \boxed{\begin{aligned} \sin'(x) &= \cos(x) \\ \cos'(x) &= -\sin(x) \end{aligned}}$$ by similar arguement

You can use sketch to remember the sign



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Def.: f is differentiable if it is diff'able at every $x \in (a,b)$

Remark: We can also write

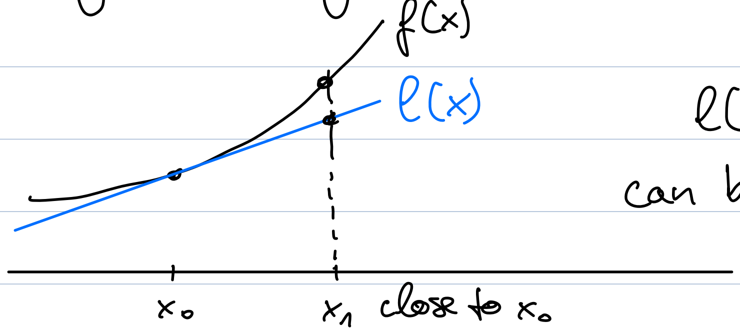
$$\boxed{f'(x) = \lim_{\xi \to x} \frac{f(\xi) - f(x)}{\xi - x}}$$

and $f'(x) = \frac{df}{dx} = \dot{f} = Df$

The derivative can be seen as the "slope of f at point x" or "instantaneous rate of change at x".

Another perspective on derivatives follows from the tangent line approximation.

We try to approximate the function f close to x by its tangent:



$$\ell(x) = f(x_0) + f'(x_0)(x - x_0)$$

can be used to approx. $f(x_1)$

$$\ell(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) \approx f(x_1)$$

Relative error of approximation by tangent line for point x_1:

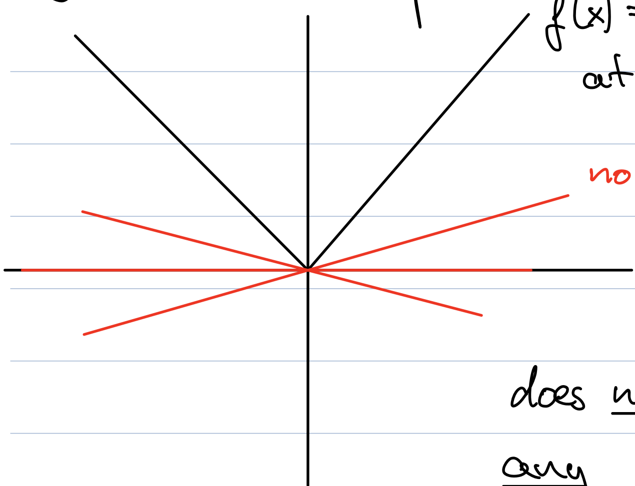
$$\frac{\overbrace{f(x_1) - \ell(x_1)}^{\text{distance in } y}}{\underbrace{x_1 - x_0}_{\text{distance in } x}} = \frac{f(x_1) - \left(f(x_0) + f'(x_0)(x_1 - x_0)\right)}{x_1 - x_0}$$

$$= \underbrace{\frac{f(x_1) - f(x_0)}{x_1 - x_0}}_{\xrightarrow[x_1 \to x_0]{} f'(x_0) \text{ if } f \text{ is diff'able at } x_0} - f'(x_0) \xrightarrow[x_1 \to x_0]{} 0 \quad \text{if } f \text{ is diff'able at } x_0$$

↪ For a diff'able function, the relative error of the linear approximation is going to zero when approaching the point (x_0) where the line is tangent to the function.

↪ A diff'able function is characterized by the fact that it can be approximated locally by a linear function

Counter example:



$f(x) = |x|$ at $x_0 = 0$

no unique tangent!

Here, relative error of approx. does <u>not</u> vanish as $x \to 0$ for <u>any</u> of the many possible tangent lines.