Calculus and Elements of Linear Algebra I   Session 9

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

2. Derivatives

2.1 Introduction to derivatives and their properties

Topic 2.1.B: Differentiation rules

Addition rule: If f', g' exist, then $\boxed{(f+g)'(x) = f'(x) + g'(x)}$

Product rule: If f', g' exist, then $\boxed{(f \cdot g)'(x) = f(x) \cdot g'(x) + f'(x) \cdot g(x)}$

Proof product rule: $(f \cdot g)'(x) = \lim\limits_{h \to 0} \dfrac{(f \cdot g)(x+h) - (fg)(x)}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \underbrace{\lim_{h \to 0} f(x+h) \frac{g(x+h) - g(x)}{h}} + g(x) \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{= f'(x)}$$

$$\underbrace{\lim_{h\to 0} f(x+h)}_{=f(x)} \underbrace{\lim_{h\to 0} \frac{g(x+h)-g(x)}{h}}_{=g'(x)}$$

because
f is continuous

$$= f(x)\cdot g'(x) + f'(x)\cdot g(x)$$

Note: If derivative of f exists at $x \in (a,b)$, then f is continuous at x.

"differentiability implies continuity"
(but not the other way around)

Example: $|x|$ at $x = 0$ is continuous but not diff'able

Diff. of constant: $c \in \mathbb{R}$ is a constant: $\boxed{c' = 0}$

Multiplication with constant: $\boxed{(c\cdot f)'(x) = c\, f'(x)}$

Quotient rule: (if $g(x) \neq 0$)

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{f'(x)\cdot g(x) - g'(x)\, f(x)}{(g(x))^2}}$$

Proof quotient rule: $g(x)\cdot\frac{f(x)}{g(x)} = f(x)$ (if $g(x)\neq 0$)

$$\Rightarrow \left(g\cdot\frac{f}{g}\right)'(x) = f'(x)$$

$$\overset{\text{Product}}{\underset{\text{rule}}{\Longrightarrow}} \quad g'(x)\cdot\left(\frac{f}{g}\right)(x) + g(x)\cdot\left(\frac{f}{g}\right)'(x) = f'(x)$$

$$\Rightarrow \left(\frac{f}{g}\right)'(x) = \frac{1}{g(x)}\left(f'(x) - g'(x)\cdot\left(\frac{f}{g}\right)(x)\right) = \frac{g(x)f'(x) - g'(x)f(x)}{(g(x))^2}$$

$\blacksquare$

Chain rule: $$\boxed{\left(f(g(x))\right)' = f'(g(x))\cdot g'(x)}$$

"outer derivative times inner derivative"

Proof chain rule: in live lecture

Power rule: $n\in\mathbb{N}$, $\boxed{(x^n)' = n\,x^{n-1}}$

Proof power rule: Proof by induction

$n=1$: $\quad x^1 = x$, $\quad x' = \lim\limits_{h\to 0}\frac{x+h-x}{h} = \lim\limits_{h\to 0}\frac{\cancel{h}^{\,1}}{\cancel{h}_{\,1}} = 1$

$$= 1\cdot x^0 = 1\cdot 1$$

$n \longrightarrow n+1$: $\quad (x^{n+1})' = (x\cdot x^n)' = 1\cdot x^n + x\cdot (x^n)'$

$$= x^n + x\cdot n\,x^{n-1}$$

$$= x^n + n\cdot x^n = (n+1)x^n$$

$\blacksquare$

$\hookrightarrow$ We can extend the power rule to rational exponents. Proof in live lecture.

$$\left(x^{\frac{p}{q}}\right)' = \frac{p}{q}\, x^{\frac{p}{q}-1} \quad , \text{ for } p,q \in \mathbb{Z}$$

It can be further extended to arbitrary $r \in \mathbb{R}$ "by continuity" (i.e. representing r as a limit of rational numbers).

$$\boxed{x^{r} = r\, x^{r-1} \quad , \; r \in \mathbb{R}}$$

Example differentiation rules:

$$f(x) = \cos(4x)$$

$$f'(x) = -\sin(4x)\cdot 4 \cdot 1 = -4\sin(4x)$$

More examples in live lecture

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Derivative of exponential function

$$(e^x)' = \underbrace{\lim_{h \to 0} \frac{e^{x+h} - e^x}{h}}_{\boxed{= \frac{e^x \cdot e^h - e^x}{h} = e^x \frac{e^h - 1}{h}}} = e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{=1}$$

(Proof more complex; Analysis I)

So in short: $\boxed{(e^x)' = e^x}$ (function and its gradient are the same)

Ex.: $(e^{5x})' = e^{5x} \cdot 5 = 5e^{5x}$

Derivative of inverse function

$f : (a,b) \longrightarrow \mathbb{R}$ diff'able, invertible on its range, with inverse $g = f^{-1}$

$$\boxed{g(f(x)) = x}$$ (definition of inverse)

Take derivative: (chain rule) $g'(f(x)) \cdot f'(x) = 1$

[Remark: From, e.g., Analysis I
f diff'able at x with $f'(x) \neq 0$
$\Rightarrow$ g diff'able at $y = f(x)$]

Solve for $g'(x)$:

$$g'(f(x)) = \frac{1}{f'(x)}, \text{ with } y = f(x)$$

$$\Rightarrow \boxed{g'(y) = \frac{1}{f'(x)} = \frac{1}{f'(g(y))}}$$

Ex.:

① $f(x) = x^a$, $a > 0$

$$f'(x) = a\,x^{a-1}, \quad g(y) = y^{1/a} = x$$

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{a x^{a-1}} \overset{x=y^{\frac{1}{a}}}{=} \frac{1}{a\left(y^{\frac{1}{a}}\right)^{a-1}}$$

$$= \frac{1}{a}\, y^{-\left(\frac{a-1}{a}\right)} = \boxed{\frac{1}{a}\, y^{\frac{1}{a}-1} = \left(y^{\frac{1}{a}}\right)'}$$

(2) $f(x) = e^x, \quad f'(x) = e^x$

$$g(y) = \ln y = x$$

$$(\ln y)' = \frac{1}{e^x} = \frac{1}{e^{\ln y}} = \frac{1}{y}$$

change notation $\boxed{(\ln(x))' = \frac{1}{x}}$

(3) $f(x) = \tan x = \frac{\sin x}{\cos x} = y$

$$f'(x) = \frac{\cos(x)\cdot\cos(x) - \sin(x)\cdot(-\sin x)}{\cos^2(x)}$$

quotient rule

$$= \frac{\overbrace{\cos^2(x) + \sin^2(x)}^{=1}}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$g(y) = \arctan y = x$ (inverse of tan)

$$(\arctan y)' = \frac{1}{f'(x)} = \cos^2 x$$

with $y = \frac{\sin x}{\cos x} \Rightarrow y^2 = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x}$

$$\Rightarrow \cos^2 x \cdot y^2 = 1 - \cos^2 x$$

$$\Rightarrow \cos^2 x \, (1 + y^2) = 1$$

$$\Rightarrow \cos^2 x = \frac{1}{1 + y^2}$$

$$\Rightarrow \boxed{(\arctan y)' = \frac{1}{1 + y^2}}$$