Dr. Stephan Juricke, Prof. Sören Petrat (based on lecture notes by Marcel Oliver)
Jacobs University, Fall 2022

Example Session for:

Topic 4.A: Common Ordinary Differential Equations

Topic 5.A: Introduction to Vectors and Vector Operations

Newton's law of Cooling:

$T(t)$ = temperature at time t

$k>0$ = rate coefficient, A = ambient temperature

ODE: $\frac{dT}{dt} = k(A-T)$ with $T(0)=T_0$

Separation of variables: $\underbrace{\int_{T_0}^{T(t)} \frac{1}{A-T}\,dT}_{= -\ln(A-T)\big|_{T_0}^{T(t)} = \ln \frac{A-T_0}{A-T(t)}} = \underbrace{\int_0^t k\,dt}_{=kt}$

$$\Rightarrow \frac{A-T_0}{A-T(t)} = e^{kt} \Rightarrow A-T(t) = (A-T_0)e^{-kt} \Rightarrow T(t) = A + (T_0 - A)e^{-kt}$$

Check: $T(0) = A + (T_0 - A) = T_0$ ✓

Note: $T(t) \xrightarrow{t\to\infty} A$, i.e., the body assumes the ambient temperature

Another Example of Separation of Variables:

$$\frac{dy}{dt} = -3yt \quad \text{with } y(0) = 1. \quad \text{(ODE is separable, but not autonomous.)}$$

$$\Rightarrow \underbrace{\int_1^{y(t)} \frac{1}{y}\, dy}_{\substack{= \ln y \,\big|_1^{y(t)} \\ = \ln y(t)}} = -3 \underbrace{\int_0^t \tilde{t}\, d\tilde{t}}_{= -\frac{3}{2}t^2} \qquad \Rightarrow y(t) = e^{-\frac{3}{2}t^2}$$

Predator-Prey Models / Lotka-Volterra Equations:

A coupled system of two ODEs:

prey: $\frac{dy}{dt} = \underbrace{by}_{\text{growth of prey } y} - \underbrace{rxy}_{\text{decline of population through predators } x}$

predator: $\frac{dx}{dt} = \underbrace{-sx}_{\text{decline of predators } x \text{ in absence of prey}} + \overbrace{cxy}^{\text{growth of } x \text{ by availability of prey } y}$

Here, we can deduce an equation relating y and x: $\frac{dy}{dx} = \frac{by - rxy}{-sx + cxy} = \frac{(b-rx)}{x} \; \frac{y}{(-s+cy)}$

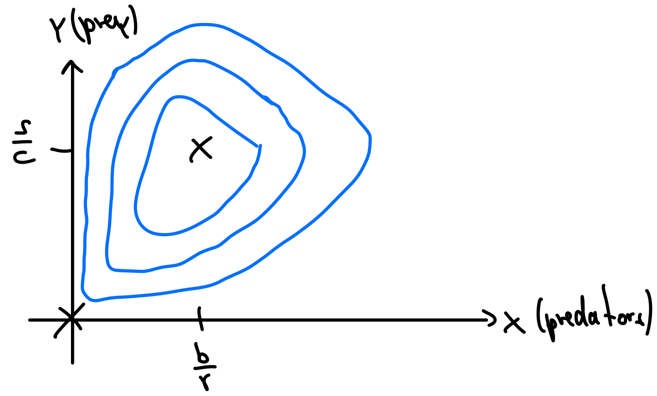
$$\Rightarrow \text{separation of variables: } \underbrace{\int \frac{-s+cy}{y}\, dy}_{\substack{= \int(-\frac{s}{y} + c)\, dy \\ = -s\ln y + cy}} = \underbrace{\int \frac{b-rx}{x}\, dx}_{= \int (\frac{b}{x} - r)\, dx = b\ln x - rx + \underbrace{\tilde{c}}}$$

here, we don't specify the integration boundaries, but we introduce an integration constant $\tilde{c}$, which is determined by initial data

$$\Rightarrow -s\ln y + cy = b\ln x - rx + \tilde{c}$$

Note: Our ODE has two equilibrium points:
- $x = 0, y = 0$
- $x = \frac{b}{r}, y = \frac{s}{c}$

A plot of x vs. y yields:



$\Rightarrow$ Populations are stable when $x = \frac{b}{r}$ and $y = \frac{s}{c}$, otherwise the populations grow/decline along one of the blue lines.

Calculus and Elements of Linear Algebra I

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

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$\boxed{\text{Live lectures sessions 19 \& 20}}$

5. Vectors & vector spaces

Examples scalar and cross product:

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

$$|u| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, \quad |v| = \sqrt{2^2 + 4^2 + 6^2} = \sqrt{56}$$

$$u \cdot v = 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 = 28$$

$$u \cdot v = |u|\,|v| \cos\theta = \sqrt{14 \cdot 56} \cos\theta = 28 \cdot \cos\theta \stackrel{!}{=} 28$$

$$\Rightarrow \theta = 0^\circ \text{ or } 180^\circ$$

Scalar product is maximum for parallel vectors

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 6 - 3 \cdot 4 \\ 3 \cdot 2 - 1 \cdot 6 \\ 1 \cdot 4 - 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$|u \times v| = |u|\,|v| \sin\theta = 28 \cdot \sin\theta \overset{!}{=} \sqrt{0^2 + 0^2 + 0^2} = 0$$

$\Rightarrow \theta = 0°$ or $180°$

Length of cross product vector
is at minimum (= 0)
for parallel vectors

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$|u| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$, $\quad |v| = \sqrt{(-1)^2 + (-1)^2 + (1)^2} = \sqrt{3}$

$$u \cdot v = 1 \cdot (-1) + 2 \cdot (-1) + 3 \cdot 1 = 0$$

$$u \cdot v = |u|\,|v| \cos\theta = \sqrt{3 \cdot 14} \cos\theta = \sqrt{42} \cdot \cos\theta = 0$$

$\Rightarrow \theta = 90°$ or $270°$

Scalar product is minimum (= 0)
for perpendicular vectors

$$u \times v = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 - 3 \cdot (-1) \\ 3 \cdot (-1) - 1 \cdot 1 \\ 1 \cdot (-1) - 2 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix} = \omega$$

$$|u \times v| = |\omega| = \sqrt{5^2 + (-4)^2 + 1^2} = \sqrt{42}$$

$$|u \times v| = |u|\,|v| \sin\theta = \sqrt{42} \cdot \sin\theta \overset{!}{=} \sqrt{42}$$

$\Rightarrow$ $\theta = 90°$ or $270°$

Length of cross product vector is at minimum ($\sin\theta = 1$) for perpendicular vectors

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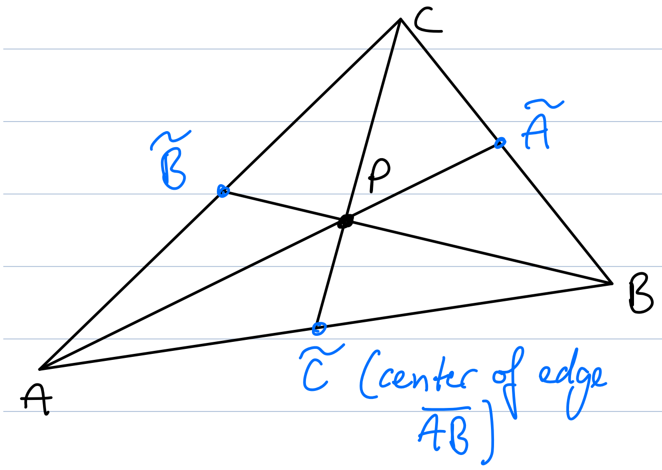
Vector application: Centroid of triangle



P is centroid

Q1: Does this construction give a single point of intersection?

Q2: What is coordinate of P

vector pointing at B

Coordinates of $\tilde{A}, \tilde{B}, \tilde{C}$: $\tilde{a} = \frac{1}{2} b + \frac{1}{2} c$

vector pointing at $\tilde{A}$

$\tilde{b} = \frac{1}{2} a + \frac{1}{2} c$

$\tilde{c} = \frac{1}{2} a + \frac{1}{2} b$

halfway between A and B $(\lambda = \frac{1}{2})$

from lecture:
point on a line segment $\overline{AB}$
$p = (1-\lambda)a + \lambda b$
$\lambda \in [0,1]$

Line segment $\tilde{A}A$: $\lambda a + (1-\lambda)\tilde{a}$

" $\tilde{B}B$: $\mu b + (1-\mu)\tilde{b}$

" $\tilde{C}C$: $\kappa c + (1-\kappa)\tilde{c}$

$\Rightarrow$ using $\tilde{a}$ and $\tilde{b}$ from above

$$\Rightarrow \lambda a + (1-\lambda)\tfrac{1}{2} b + (1-\lambda)\tfrac{1}{2} c = (1-\mu)\tfrac{1}{2} a + \mu b + (1-\mu)\tfrac{1}{2} c \qquad (*)$$

To solve $(*)$, let's equate coefficients in front of a, b, c separately:

$$\lambda = \frac{1-\mu}{2}, \quad \frac{1-\lambda}{2} = \mu, \quad \frac{1-\lambda}{2} = \frac{1-\mu}{2} \Rightarrow \lambda = \mu$$

All equations are consistent ($\exists$ exactly one solution)

$$\Rightarrow \lambda = \frac{1-\lambda}{2} \Rightarrow 2\lambda = 1 - \lambda \Rightarrow \lambda = \frac{1}{3} = \mu$$

$$p = \frac{1}{3}a + \left(1-\frac{1}{3}\right)\cdot\frac{1}{2}b + \left(1-\frac{1}{3}\right)\cdot\frac{1}{2}c = \frac{1}{3}(a+b+c)$$

Using $\overline{C\tilde{C}}$ and $\overline{B\tilde{B}}$ (or $\overline{A\tilde{A}}$) to do this computation will give the same point by symmetry!