Calculus and Elements of Linear Algebra I

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Live lectures sessions 21 & 22

5. Vectors & vector spaces

and

6. Matrices

6.1 Introduction to matrices and link to linear operators

Vector spaces

Examples: • $V = \{$continuous functions on $[0,1]\}$

Non-examples: • images with greyscale values $0 \dots 255$
(e.g. not closed under addition)

- probability vectors: $p = (p_1, \ldots, p_n)$, $p_i \in [0,1]$

$$p_1 + \ldots + p_n = 0$$

(e.g. violated by scalar multiplication)

- $\mathbb{R}^3$ where "addition" is the cross product.

(e.g. not commutative)

Remember:

$\boxed{\text{Theorem: } B \text{ is a basis} \Rightarrow \text{all vectors in } B \text{ are linearly independent}}$

Proof: Suppose not. Then there exist $\alpha_1, \ldots, \alpha_n$ not all zeros s.t.

$$0 = \sum \alpha_i b_i$$

but also

$$0 = 0 \cdot b_1 + \ldots + 0 \cdot b_n$$

This contradicts uniquness of representation of 0.

Matrices: Take example from session 22

V = vector space of polynomials of degree ≤ 2 with coefficients in V in $\mathbb{R}$

$\mathcal{A} = \frac{d}{dx}$ differential operator

But we use $C = \{x^2-1, x^2+1, x\}$ as basis of V

instead of $B = \{1, x, x^2\}$

To get $p = 3x^2 - 2x + 7$ in this basis, we need to write:

$$\alpha_1 (x^2-1) + \alpha_2 (x^2+1) + \alpha_3 x = 3x^2 - 2x + 7$$

$$(\alpha_1 + \alpha_2) x^2 + (\alpha_2 - \alpha_1) \cdot 1 + \alpha_3 x = 3x^2 - 2x + 7$$

Compute coefficients:

$$\left.\begin{array}{l} \alpha_1 + \alpha_2 = 3 \\ \alpha_2 - \alpha_1 = 7 \end{array}\right\} 2\alpha_2 = 10 \Rightarrow \alpha_2 = 5 \quad \Rightarrow \alpha_1 = -2$$

$$\alpha_3 = -2$$

Here: $\alpha = \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$

$$\mathcal{A}(x^2-1) = \frac{d}{dx}(x^2-1) = 2x = 0 \cdot (x^2-1) + 0 \cdot (x^2+1) + 2 \cdot x$$

$$\mathcal{A}(x^2+1) = 2x = 0 \cdot (x^2-1) + 0 \cdot (x^2+1) + 2 \cdot x$$

$$\mathcal{A}(x) = 1 = -\frac{1}{2}(x^2-1) + \frac{1}{2}(x^2+1) + 0 \cdot x$$

$$\Rightarrow \qquad A = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix}, \qquad A\alpha = \begin{pmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 5 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \cdot -2 + 0 \cdot 5 + \left(-\frac{1}{2}\right) \cdot (-2) \\ 0 \cdot -2 + 0 \cdot 5 + \frac{1}{2} \cdot (-2) \\ 2 \cdot -2 + 2 \cdot 5 + 0 \cdot (-2) \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 6 \end{pmatrix}$$ as a vector.

This represents $1 \cdot (x^2 - 1) + (-1) \cdot (x^2 + 1) + 6x = 6x - 2$

$= \frac{dp}{dx}$ ✓