Calculus and Elements of Linear Algebra I

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Live lectures sessions 23 & 24

6. Matrices

6.1 Introduction to matrices and link to linear operators

and

6.2 Solving systems of linear equations

Matrix-matrix multiplication:

$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 5 \end{pmatrix}$$

Solving systems of linear equations:

We want to solve $Ax = b$ with

$$\begin{pmatrix} 2 & -2 & -4 & -1 & 5 \end{pmatrix} \qquad \begin{pmatrix} 5 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -3 & -6 & 3 & 12 \\ 3 & -3 & -6 & 2 & 11 \end{pmatrix}, \quad b = \begin{pmatrix} 21 \\ 18 \end{pmatrix}$$

Here, $x \in \mathbb{R}^5$ as $A \in M(3 \times 5)$, i.e. we have more unknowns than equations (more columns than rows)

(This is a so called underdetermined system, see next session, i.e. it is underconstrained)

Augmented matrix:

$$\left(\begin{array}{ccccc|c} 2 & -2 & -4 & -1 & 5 & 5 \\ 3 & -3 & -6 & 3 & 12 & 21 \\ 3 & -3 & -6 & 2 & 11 & 18 \end{array}\right) \xrightarrow[R1 \to R2]{R2/3 \to R1} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 4 & 7 \\ 2 & -2 & -4 & -1 & 5 & 5 \\ 3 & -3 & -6 & 2 & 11 & 18 \end{array}\right)$$

$$\xrightarrow[R3-3R1 \to R3]{R2-2R1 \to R2} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 1 & 4 & 7 \\ 0 & 0 & 0 & -3 & -3 & -9 \\ 0 & 0 & 0 & -1 & -1 & -3 \end{array}\right)$$

$$\xrightarrow[\frac{R2}{-3} \to R2,\ R3 - \frac{R2}{3} \to R3]{R1+R3 \to R1} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

← one equation does not contain additional information, i.e. $0=0$

Now, we fill in the matrix with additional free variables:

$$\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\text{to diagonal}]{\text{reorder according}} \left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

are called <u>pivots</u>

"missing pivots

(more about pivots next week)

Solution is now:

linearly independent solutions to the homogeneous problem $Ax = 0$

$$x = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

Solution for inhomogeneous problem from last column

1^{st} column with missing pivot. Set pivot to -1

2^{nd} column with missing pivot. Set pivot to -1

3^{rd} column with missing pivot. Set pivot to -1

Any such x with $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ is a solution to $Ax = b$