Calculus and Elements of Linear Algebra I

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

$\boxed{\text{Live lectures sessions 25 \& 26}}$

6. Matrices

6.2 Solving systems of linear equations

From last week's example lecture:
We want to solve $Ax = b$ with

$$A = \begin{pmatrix} 2 & -2 & -4 & -1 & 5 \\ 3 & -3 & -6 & 3 & 12 \\ 3 & -3 & -6 & 2 & 11 \end{pmatrix}, \quad b = \begin{pmatrix} 5 \\ 21 \\ 18 \end{pmatrix}$$

and we get $\left(\begin{array}{ccccc|c} 1 & -1 & -2 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ after Gaussian elimination

with two pivots.
Rank-nullity theorem then states for $A \in M(3 \times 5)$:

$\operatorname{rank} A = 2$ and $m = 5$

so

$$\text{nullity } A + \underbrace{\operatorname{rank} A}_{=2} = \underbrace{m}_{=5}$$

$\Rightarrow$ nullity $A = 3$, i.e. we have a 3D null space or kernel

We got the solution last week:

$$x = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 3 \\ 0 \end{pmatrix} + \lambda_1 \underbrace{\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + \lambda_2 \underbrace{\begin{pmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} + \lambda_3 \underbrace{\begin{pmatrix} 3 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}}_{v_3}$$

where v_1, v_2, v_3 are basis vectors to the null space for which $Av = 0$.

<u>Change of basis:</u> Let us continue with the example of change of basis from session 26.

First basis to vector space of polynomials of degree ≤ 2 was $E = \{\underset{e_1}{1}, \underset{e_2}{z}, \underset{e_3}{z^2}\}$

Second basis was $C = \{\underbrace{2z-1}_{b_1}, \underbrace{2z+1+z^2}_{b_2}, \underbrace{z^2-1}_{b_3}\}$

using z as variable here

To transform coordinates $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ of polynomial $p(z)$ w.r.t. e_1, e_2, e_3 to coordinates y w.r.t. b_1, b_2, b_3 we had to solve:

$$Sy = x \quad , \quad S = \begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

coordinate vector for b_1 w.r.t. basis E

$$\Rightarrow \quad y = S^{-1} x$$

We got

$$S^{-1} = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \quad \text{by Gaussian elimination}$$

Now, let $p(z) = 2z^2 - z + 4$ with coordinates w.r.t. e_1, e_2, e_3

$$x = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} \quad , \quad \frac{dp}{dz} = p'(z) = 4z - 1$$

Then we get the coordinates w.r.t. basis b_1, b_2, b_3:

$$y = S^{-1} x = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{7}{3} \\ \frac{11}{6} \\ \frac{1}{6} \end{pmatrix}$$

So: $p(z) = -\frac{7}{3}(2z - 1) + \frac{11}{6}(2z + 1 + z^2) + \frac{1}{6}(z^2 - 1)$

$= -z + 4 + 2z^2$ ✓

<u>Recall:</u> Differentiation is represented w.r.t. e_1, e_2, e_3 by

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{(Session 22, part 2)}$$

$$e_1' = 0 \quad e_2' = 1 \quad e_3' = 2z$$

<u>Q:</u> What matrix represents differentiation w.r.t. b_1, b_2, b_3?

Since

$$y' = S^{-1}x' = S^{-1}\underbrace{Dx}_{=x'} = S^{-1}D\underbrace{S\,S^{-1}}_{=I}x = S^{-1}DSy$$

$$S^{-1}DS = \underbrace{\begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}\begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{2}{3} & -1 & -\frac{1}{3} \end{pmatrix}$$

$$S^{-1}DSy = \begin{pmatrix} -\frac{2}{3} & 0 & \frac{2}{3} \\ \frac{2}{3} & 1 & \frac{1}{3} \\ -\frac{2}{3} & -1 & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} -\frac{7}{3} \\ \frac{11}{6} \\ \frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{14}{9} + \frac{1}{9} \\ -\frac{14}{9} + \frac{11}{6} + \frac{1}{18} \\ \frac{14}{9} - \frac{11}{6} - \frac{1}{18} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\Rightarrow \quad p'(z) = \frac{5}{3}\left(2z-1\right) + \frac{1}{3}\left(2z+1+z^2\right) + \left(-\frac{1}{3}\right)\left(z^2-1\right)$$

(w.r.t. b_1, b_2, b_3)

$$= 4z - 1 \quad \checkmark$$

We can sketch this in an abstract way:

$$\begin{tikzcd}
(V, E) \arrow[r, "D"] \arrow[d, dashed, shift right, "S^{-1}"'] & (V, E) \arrow[d, dashed, shift right, "S^{-1}"'] \\
(V, C) \arrow[r, "T"] \arrow[u, shift right, "S"'] & (V, C) \arrow[u, shift right, "S"']
\end{tikzcd}$$

vector space   basis

So, $\quad T = SDS^{-1}$

S changes from basis C to E and S^{-1} from E to C

D and T are operators (e.g. differentiation) on V w.r.t. to the same basis, E and C, respectively.

Important topics for next semester as continuation of Linear Algebra:

<u>Further properties of matrices:</u>

Determinant of a matrix A, $\det(A)$
and
Eigenvalues and eigenvectors, i.e.

$$A\underbrace{v}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} v$$