Calculus and Elements of Linear Algebra I — Example Session Week 1

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Example Session for:

Topic 1.1.A: Numbers and Roots of Polynomials

Topic 1.1.B: Complex Numbers and the Fundamental Theorem of Algebra

Numbers:

$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

In general:
- $A \subset B$ means: $\forall a \in A$ we have $a \in B$.
- $\emptyset := \{\}$ the empty set.
- $A \cup B = \{x : x \in A \text{ or } x \in B\}$. E.g., $\{1,2,3\} \cup \{2,3,4\} = \{1,2,3,4\}$.
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$. E.g., $\{1,2,3\} \cap \{2,3,4\} = \{2,3\}$.

We call $a = a_1 \cdots a_n$, with $a_1, \ldots, a_n$ prime numbers, a factorization of a.

E.g., $20 = 2 \cdot 2 \cdot 5$, $87 = 3 \cdot 29$.

Complex Numbers:

$\mathbb{C} \ni z = x + iy$ with $x, y \in \mathbb{R}$, and i the imaginary unit ($i^2 = -1$).

- Complex conjugation: $(x+iy)^* = x - iy$.

  E.g., $(5+3i)^* = 5-3i$.

- Multiplication: $(x+iy)\cdot(a+ib) = xa + x(ib) + (iy)a + (iy)(ib) = xa - yb + i(xb + ya)$.

  E.g., $(1+2i)(3+4i) = 3 - 8 + i(4+6) = -5 + 10i$.

  E.g., $i^3 = i^2 i = -i$, $i^4 = i^2 i^2 = (-1)(-1) = 1$, $i^5 = i^4 \cdot i = i$, $i^6 = -1$, ...

- Division: $\frac{x+iy}{a+ib} = \frac{(x+iy)(a-ib)}{(a+ib)(a-ib)} = \frac{xa + yb + i(-xb + ya)}{a^2 + b^2} = \frac{xa + yb}{a^2+b^2} + i\left(\frac{-xb + ya}{a^2+b^2}\right)$.

  E.g., $\frac{2+3i}{1+i} = \frac{(2+3i)(1-i)}{(1+i)(1-i)} = \frac{5+i}{2} = \frac{5}{2} + i\frac{1}{2}$.

Note:
- $\overbrace{|z|}^{\text{absolute value}} := \sqrt{z \cdot z^*} = \sqrt{(x+iy)(x-iy)} = \sqrt{x^2 + y^2} \in \mathbb{R}$
- $(z_1 z_2)^* = \left((x_1 + iy_1)(x_2 + iy_2)\right)^* = \left(x_1x_2 - y_1y_2 + i[x_1y_2 + x_2y_1]\right)^*$

  $= x_1x_2 - y_1y_2 - i[x_1y_2 + x_2y_1] = z_1^* z_2^*$

Roots of Quadratic Equations:

$$x^2+px+q=0 \iff x_\pm = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2-q} \qquad (\text{"p-q formula"})$$

E.g.: • $x^2+3x-1=0$

$$\Rightarrow x_\pm = -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2-(-1)} = -\frac{3}{2} \pm \sqrt{\frac{9}{4}+1} = -\frac{3}{2} \pm \sqrt{\frac{13}{4}} = -\frac{3}{2} \pm \frac{\sqrt{13}}{2}$$

$$\Rightarrow x_+ = \frac{1}{2}\left(-3+\sqrt{13}\right),\ x_- = \frac{1}{2}\left(-3-\sqrt{13}\right) \qquad (\text{two real roots}).$$

$$\Rightarrow x^2+3x-1 = (x-x_+)(x-x_-) = \left(x-\tfrac{1}{2}(-3+\sqrt{13})\right)\left(x-\tfrac{1}{2}(-3-\sqrt{13})\right).$$

• $x^2+4x+4=0$

$$\Rightarrow x_\pm = -\frac{4}{2} \pm \sqrt{\left(\frac{4}{2}\right)^2-4} = -2 \pm \sqrt{4-4} = -2 \qquad (\text{one real root}).$$

$$\Rightarrow x^2+4x+4 = (x+2)^2.$$

• $x^2+2x+5=0$

$$\Rightarrow x_\pm = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right)^2-5} = -1 \pm \sqrt{1-5} = -1 \pm \sqrt{4\cdot(-1)} = -1 \pm 2i$$

$$\Rightarrow x^2+2x+5 = (x-(-1+2i))(x-(-1-2i)) = (x+1-2i)(x+1+2i) \qquad (\text{two complex roots}).$$

$$\left(\text{Check: } (x+1-2i)(x+1+2i) = x^2 + x(1+2i+1-2i) + (1-2i)(1+2i) = x^2+2x+5 \ \checkmark\right)$$

Roots of Polynomials of Degree $n>2$:

Formulas for $n=3,4$ complicated, general formulas for $n\geq 5$ don't exist.

$\Rightarrow$ Need numerical techniques such as Newton's method (later).

Often, one root can be guessed, e.g. x^3-2x^2-5x+6 has a root 1, since $1-2-5+6=0$.

$$\Rightarrow x^3-2x^2-5x+6=(x-1)\underbrace{(x^2+ax+b)}_{\text{some polynomial with unknown } a,b}.$$

But now we can compare coefficients:

$$x^3-2x^2-5x+6 = x^3+(a-1)x^2+(b-a)x-b \qquad \Rightarrow a=-1,\ b=-6$$

$b=-6$

$a-1=-2=-2$ ✓ $\qquad b-a=-6-a=-5 \Rightarrow a=-1$

Now: roots of x^2-x-6 are $x_\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4}+6} = \frac{1}{2} \pm \sqrt{\frac{25}{4}} = \frac{1}{2} \pm \frac{5}{2} = \begin{cases} 3 & \text{for } + \\ -2 & \text{for } - \end{cases}$

$$\Rightarrow x^3-2x^2-5x+6=(x-1)(x-3)(x+2).$$