Calculus and Elements of Linear Algebra I — Example Session Week 2

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Example Session for:

$\boxed{\text{Topic 1.1.C: Misc – Polynomial Long Division, Inequalities, Binomial Coefficients}}$

$\boxed{\text{Topic 1.2.A: Equations, Functions and their Inverses, Graphs}}$

Polynomial Long Division:

Example: $x^3-2x^2-5x+10$.

Guesses for roots: 1? ↯ -1? ↯ 2? Yes, since $2^3-2\cdot 2^2-5\cdot 2+10=8-8-10+10=0$.

$$\Rightarrow \quad x^3-2x^2-5x+10=(x-2)\underbrace{(x^2-5)}_{=(x-\sqrt{5})(x+\sqrt{5})}$$

$$\begin{array}{rl} -\ & x^3-2x^2 \\ \hline & 0-5x+10 \\ -\ & -5x+10 \\ \hline & 0 \end{array}$$

Example: x^3+2x^2-x-2.

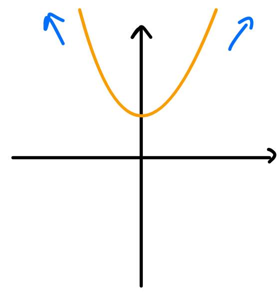
Guesses for roots: 1? Yes, since $1+2-1-2=0$.

$$\Rightarrow \quad x^3+2x^2-x-2=(x-1)\underbrace{(x^2+3x+2)}_{=(x+2)(x+1)}$$

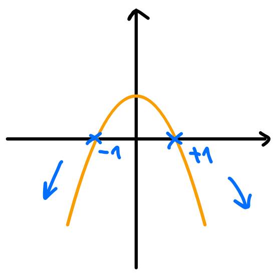
$$\begin{array}{rl} -\ & x^3-x^2 \\ \hline & 3x^2-x \\ -\ & 3x^2-3x \\ \hline & 2x-2 \\ -\ & 2x-2 \\ \hline & 0 \end{array}$$

Drawing Polynomials:

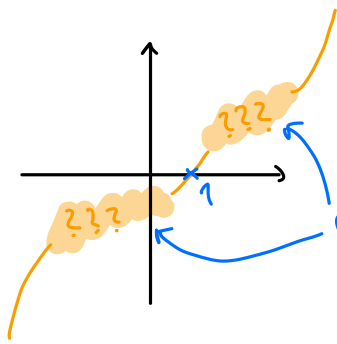
Example: $p(x) = x^2+1$ has roots $z_\pm = \pm i$, i.e., no real roots



Example: $p(x) = -x^2+1$ has roots $z_\pm = \pm 1$.



Example $p(x) = (x-1)\underbrace{(x+i)(x-i)}_{x^2+1} = x^3 - x^2 + x - 1$ has one real, two complex roots.



don't know exact behavior here (need derivatives, later)

Binomial Coefficients

Recall: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$, with $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

Example: $\binom{5}{2} = \dfrac{5!}{2!(5-2)!} = \dfrac{5\cdot 4\cdot 3\cdot 2}{2\cdot(3\cdot 2)} = 10$, i.e., $(a+b)^5 = a^5 + \dots + \underbrace{10}_{\binom{5}{2}} \overbrace{a^3 b^2}^{a^{n-k}b^k \text{ for } n=5,\,k=2} + \dots + b^5$.

Note: Pascal triangle property:

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$$

$$= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!\,k}{k!(n-k+1)!}$$

$$= \frac{n!(n+1) - n!\,k + n!\,k}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n+1-k)!}$$

$$= \binom{n+1}{k}$$

Exponential Function:

Let $a>0$. For $n\in\mathbb{N}$, we have $a^n = \underbrace{a\cdot a\cdot a\cdots a}_{n\text{ times}}$.

Important equation: $a^{n+m} = a^n a^m$. (*) (Note: this implies $(a^n)^m = \underbrace{a^n\cdots a^n}_{m\text{ times}} = a^{nm}$.)

Goal: define a^x for $x\in\mathbb{R}$.

- For $n\in\mathbb{Z}$: $1 = a^0 = a^{n-n} \overset{\text{by (*)}}{=} a^n a^{-n} \Rightarrow \boxed{a^{-n} = \frac{1}{a^n}}$.

- For $r=\frac{p}{q}\in\mathbb{Q}$: $a = a^{\frac{1}{q}q} \overset{\text{by (*)}}{=} (a^{\frac{1}{q}})^q \Rightarrow a^{\frac{1}{q}} = \sqrt[q]{a}$ (the q-th root of a).

  $$\Rightarrow \boxed{a^r = (\sqrt[q]{a})^p}.$$

- For $x\in\mathbb{R}$: The standardized way of defining a^x goes via the exponential function

  $\exp:\mathbb{R}\to(0,\infty)$, $x\mapsto e^x$ with $e=2.718\ldots$ Euler's number.

  (We will see later why e is special.)

  exp is invertible, and the inverse is called $\ln$, the natural logarithm, i.e., $e^{\ln x}=x$.

  Let us call $a=e^y$, i.e., $y=\ln a$. Then $a^x = (e^y)^x = e^{xy} = e^{x\ln a}$.

  $$\Rightarrow \boxed{a^x = e^{x\ln a}}$$