Calculus and Elements of Linear Algebra I — Example Session Week 3

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Example Session for:

Topic 1.3.A: Definition of Limits and Limit Laws

Topic 1.3.B: Asymptotes and Limits of the Exponential Function

ε-δ-exercise

Let $f(x)=\frac{1}{x}$, $x_0 \neq 0$, then the definition of $\lim\limits_{x\to x_0} \frac{1}{x}$ should yield $\frac{1}{x_0}$.

For given ε, how do we choose δ?

We want $\left|\frac{1}{x}-\frac{1}{x_0}\right| < \varepsilon$, so reformulate this to $|x-x_0| < g(x_0,\varepsilon) =: \delta$.

Let's assume $0 < x < x_0$. Then $\left|\frac{1}{x}-\frac{1}{x_0}\right| = \left|\frac{x_0-x}{x x_0}\right| < \varepsilon \iff x_0 - x < \underbrace{x x_0 \varepsilon}_{\substack{=(x-x_0+x_0)x_0\varepsilon \\ =(x-x_0)x_0\varepsilon + x_0^2\varepsilon}}$

$$\iff \underbrace{x_0 - x + (x_0-x)x_0\varepsilon}_{=(x_0-x)(1+x_0\varepsilon)} < x_0^2\varepsilon$$

$$\iff x_0 - x < \frac{x_0^2\varepsilon}{1+x_0\varepsilon} =: \delta$$

Note: this δ also works for $x > x_0$. (check this)

To summarize: Given $\varepsilon > 0$ $\exists\, \delta := \frac{x_0^2\varepsilon}{1+x_0\varepsilon}$ s.t. $\forall x$ with $|x-x_0| < \delta$ we have $\left|f(x) - \frac{1}{x_0}\right| < \varepsilon$.

Limit Laws:

$$\lim_{t\to 0} \frac{\sqrt{t+25}-5}{t} = \lim_{t\to 0} \frac{(\sqrt{t+25}-5)(\sqrt{t+25}+5)}{t\,(\sqrt{t+25}+5)} = \lim_{t\to 0} \frac{t+25-25}{t(\sqrt{t+25}+5)}$$

$$= \lim_{t\to 0} \frac{1}{\sqrt{t+25}+5} = \frac{1}{\lim_{t\to 0}\sqrt{t+25}+5} = \frac{1}{5+5} = \frac{1}{10}.$$

use limit laws (i) and (iii)

$\lim_{t\to 0}\sqrt{t+25} = \sqrt{25}$ will follow from continuity (see Session 7)

Asymptotes:

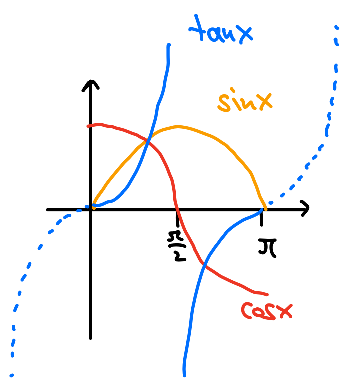
- $$\lim_{x\to\infty} \frac{x^2+3x-2}{3x^2-2x+5} = \lim_{x\to\infty} \frac{1+\frac{3}{x}-\frac{2}{x^2}}{3-\frac{2}{x}+\frac{5}{x^2}} := \lim_{y\searrow 0} \frac{1+3y-2y^2}{3-2y+5y^2} = \frac{1}{3}$$

by definition

limit law (iii)

- $$\lim_{x\nearrow\frac{\pi}{2}} \tan x = \lim_{x\nearrow\frac{\pi}{2}} \frac{\sin x}{\cos x} = \infty$$

$$\lim_{x\searrow\frac{\pi}{2}} \tan x = -\infty$$



Induction:

We claim that $\sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

- Step 1: Check for $n=1$:
  - $\sum_{k=1}^{1} k^2 = 1$
  - $\frac{1}{6} 1(1+1)(2+1) = \frac{2 \cdot 3}{6} = 1$ ✓

- Step 2: Assuming the claim holds for n, we try to show that it holds for $n+1$:

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2 = \frac{1}{6} n(n+1)(2n+1) + (n+1)^2$$

by our induction assumption

$$= \frac{1}{6}(n+1)\underbrace{\left[n(2n+1) + 6(n+1)\right]}_{= 2n^2 + n + 6n + 6 = (n+2)(2n+3)}$$

$$= \frac{1}{6}(n+1)\left[(n+1)+1\right]\left[2(n+1)+1\right],$$ which is the claim for $n+1$. ✓

$\Rightarrow$ Claim is true.

Further induction examples:

- Prove that $\sum_{k=1}^{n} (-1)^k k^2 = (-1)^n \frac{n(n+1)}{2}$ for $n \geq 1$.

- Prove that $n! > 2^n$ for $n \geq 4$.