Calculus and Elements of Linear Algebra I — Example Session Week 4

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Example Session for:

Topic 1.3.C: Continuity and the Intermediate Value Theorem

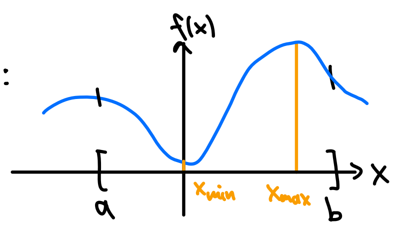
Topic 2.1.A: General Definition (of Derivatives)

## Extreme Value Theorem:

Recall the theorem:

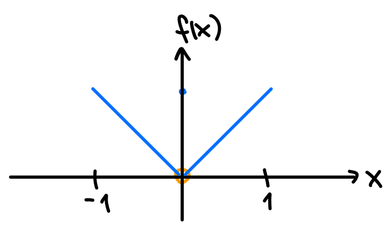
If $f:[a,b]\to\mathbb{R}$ is continuous, then $f$ assumes its minimum and maximum.

E.g.: • Theorem applies:




• Theorem does not apply bc. $f$ not continuous:

$$f(x)=\begin{cases}|x| & \text{for } x\in[-1,1],\\ 1 & \text{for } x=0.\end{cases}$$

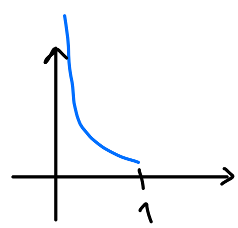



Here, the maximum 1 is assumed at $x=-1,0,1$, but the minimum is not.

• Theorem does not apply bc. interval not closed:

$f(x)=\frac{1}{x}$ with domain $(0,1)$.

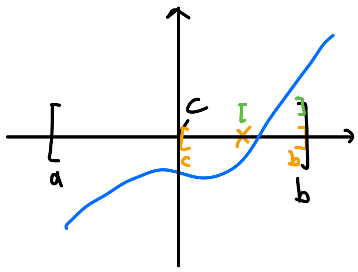
$\Rightarrow$ No maximum.



## Bisection Method:

Suppose $f : [a,b] \to \mathbb{R}$ is continuous, and $f(a) < 0$, $f(b) > 0$ (or the other way around).
Then by the intermediate value theorem, $f$ has a root in $[a,b]$.

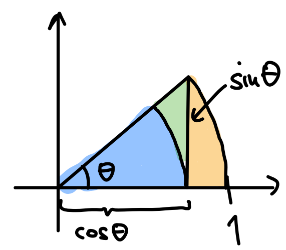
Then:
- Check $f(c)$, with $c$ the midpoint of the interval, i.e., $c = \frac{a+b}{2}$.
- If $f(c) < 0$, $\exists$ root in $[c,b]$; if $f(c) > 0$, $\exists$ root in $[a,c]$.
- Repeat until necessary precision is reached.



# Application of Squeeze Law:

Let $f(\theta) = \frac{\sin\theta}{\theta}$. What is $\lim_{\theta \to 0} f(\theta)$ ?

Consider the following picture:




The areas are: $A_{b+g} = \frac{1}{2}\cos\theta\sin\theta$ (blue + green)

$$A_b = \underbrace{\frac{\theta}{2\pi}}_{\text{fraction of the circle, "cake piece"}} \underbrace{\pi r^2}_{\text{area of circle with radius } r} = \frac{1}{2}\theta\cos^2\theta \quad \text{(blue)}$$

$$A_{b+g+o} = \frac{1}{2}\theta \quad \text{(blue + green + orange)}$$

We have $A_b \leq A_{b+g} \leq A_{b+g+o}$, i.e., $\frac{1}{2}\theta\cos^2\theta \leq \frac{1}{2}\cos\theta\sin\theta \leq \frac{1}{2}\theta$

$$\Rightarrow \underbrace{\cos\theta}_{\to 1 \text{ as } \theta \to 0} \leq \frac{\sin\theta}{\theta} \leq \underbrace{\frac{1}{\cos\theta}}_{\to 1 \text{ as } \theta \to 0}$$

$$\Rightarrow \boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.}$$

## The Exponential Function Again:

Let us define $\exp(x) := \lim\limits_{n\to\infty} \left(1+\frac{x}{n}\right)^n$.

Then:

$\frac{a}{n}=x,\ \frac{b}{m}=y$

$$\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n \lim_{m\to\infty}\left(1+\frac{b}{m}\right)^m = \lim_{x\to 0} \underbrace{(1+x)^{\frac{a}{x}}}_{=\left[(1+x)^{\frac{1}{x}}\right]^a} \lim_{y\to 0} \underbrace{(1+y)^{\frac{b}{y}}}_{=\left[(1+y)^{\frac{1}{y}}\right]^b} = \left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]^{a+b}$$

assuming $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}$ exists and using continuity of $y^a, y^b$

$$= \lim_{x\to 0}(1+x)^{\frac{a+b}{x}}$$

$\frac{a+b}{x}=n$

$$= \lim_{n\to\infty}\left(1+\frac{a+b}{n}\right)^n$$

$\Rightarrow \exp(a)\exp(b) = \exp(a+b)$

Thus $\exp(x) = e^x$ with $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.72\ldots$

## Definition of Derivative:

We can reformulate the definition in a way that is closer to the idea of linear approximation.

**Definition:**

$f:(a,b) \longrightarrow \mathbb{R}$ is differentiable at $x \in \mathbb{R}$ if $\exists\, m \in \mathbb{R}$ s.t.

$$\underbrace{f(x+h)}_{\text{f evaluated close to x}} = \underbrace{f(x)}_{\text{f at x}} + \underbrace{m \cdot h}_{\text{linear approximation}} + \underbrace{E_x(h)}_{\text{error that goes to 0 faster than linear}} \quad \text{with} \quad \frac{E_x(h)}{h} \xrightarrow{h \to 0} 0.$$

If such an $m$ exists, then $m = \dfrac{f(x+h)-f(x)}{h} + \underbrace{\dfrac{E_x(h)}{h}}_{\to 0 \text{ as } h \to 0}$

$$\Rightarrow \quad m = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = f'(x).$$