Calculus and Elements of Linear Algebra I

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Live lectures sessions 9 & 10

2. Derivatives

2.1 Introduction to derivatives and their properties

Examples:

1. $f(x) = \dfrac{3x^2+1}{x^5+x}$

Two options: Product or quotient rule

a) $f(x) = \underbrace{(3x^2+1)}_{=g(x)}\underbrace{(x^5+x)^{-1}}_{h(x)}$

$g'(x) = 6x$; $h'(x) = -(x^5+x)^{-2}\cdot(5x^4+1)$

(chain rule)

$$\Rightarrow \quad f'(x) = g'(x)\,h(x) + g(x)\,h'(x)$$

$$= 6x\,(x^5+x)^{-1} + (3x^2+1)\cdot -(x^5+x)^{-2}(5x^4+1)$$

$$= \frac{6x \cdot (x^5+x)}{(x^5+x)\cdot(x^5+x)} - \frac{(3x^2+1)(5x^4+1)}{(x^5+x)^2}$$

$$= \boxed{\frac{6x(x^5+x) - (3x^2+1)(5x^4+1)}{(x^5+x)^2}}$$

or

b) $f(x) = \dfrac{\overbrace{3x^2+1}^{=g(x)}}{\underbrace{(x^5+x)}_{=h(x)}}$

$g'(x) = 6x \qquad ; \qquad h'(x) = 5x^4 + x$

$$f'(x) = \frac{g'(x)\,h(x) - g(x)\,h'(x)}{(h(x))^2}$$

$$= \boxed{\frac{6x(x^5+x) - (3x^2+1)(5x^4+x)}{(x^5+x)^2}}$$

(you can potentially simplify)

further

② $f(x) = \sin(\cos(x))$

Use chain rule:

$$\boxed{f'(x) = \underbrace{\cos(\cos(x))}_{\text{outer}} \cdot \underbrace{(-\sin(x))}_{\text{inner}}}$$

③ $f(x) = \sin(\cos(x^2 + e^x))$

Use chain rule twice:

$$f'(x) = \underbrace{\cos(\cos(x^2+e^x))}_{\text{1st outer}} \cdot \underbrace{(\cos(x^2+e^x))'}_{\text{1st inner}}$$

$$\boxed{f'(x) = \underbrace{\cos(\cos(x^2+e^x))}_{\text{1st outer}} \cdot \underbrace{-\sin(x^2+e^x)}_{\text{2nd outer}} \cdot \underbrace{(2x+e^x)}_{\text{2nd inner}}}$$

④ $f(x) = \underbrace{e^{\cos(x)}}_{=g(x)} \cdot \underbrace{\sin(x^2 \cdot \ln(x))}_{=h(x)}$

Use product and chain rule:

$$g'(x) = \underbrace{e^{\cos(x)}}_{\text{outer}} \underbrace{(-\sin(x))}_{\text{inner}}$$

$$h'(x) = \underbrace{\cos(x^2 \cdot \ln(x))}_{\text{outer}} \underbrace{\left(2x \cdot \ln(x) + x^2 \tfrac{1}{x}\right)}_{\text{inner}}$$

$$= \cos(x^2 \cdot \ln(x)) \left(x\left(2\ln(x) + 1\right)\right)$$

$$\Rightarrow f'(x) = g'(x)\, h(x) + g(x)\, h'(x)$$

$$= \boxed{e^{\cos(x)} \cdot -\sin(x) \cdot \sin(x^2 \cdot \ln(x)) + e^{\cos(x)} \cdot \cos(x^2 \cdot \ln(x)) \cdot x\left(2\ln(x) + 1\right)}$$

Some missing proofs:

Chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Proof:

$$(f(g(x)))' = \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \underbrace{\frac{g(x+h) - g(x)}{h}} \right)$$

$= 1$

$$\xrightarrow[h\to 0]{} g(x)$$

$$\frac{f(g(x)+k)-f(g(x))}{k} = \phi(k) \quad \text{with } k := g(x+h)-g(x)$$

If k is an arbitrary, non-zero number, then

$$\lim_{k\to 0} \phi(k) = f'(g(x))$$

Moreover, by continuity of $g(x)$:

$$\lim_{h\to 0} k(h) = \lim_{h\to 0} \big(g(x+h)-g(x)\big) = 0$$

So $h \to 0$ implies $k(h) \to 0$

This proof is almost complete, but we have to be careful in case $k(h)$ becomes 0.

Solution: Extend $\phi(k)$ continuously by $f'(g(x))$ for $k=0$.

Extension of power rule $(x^n)' = n x^{n-1}$, $n \in \mathbb{N}$ to rational exponents:

$$\left(x^{\frac{p}{q}}\right)' = \left(\left(u(x)\right)^p\right)' \quad , \; u(x) = x^{\frac{1}{q}}$$

$$= \underbrace{p\,u^{p-1}}_{\text{outer}} \cdot \underbrace{u'}_{\text{inner}} \quad (*) \quad \text{(chain rule)}$$

Also: $u(x) = x^{\frac{1}{q}} \Rightarrow (u(x))^q = x$

$$\Rightarrow \underbrace{q\,u(x)^{q-1} \cdot u'}_{\text{chain rule}} = \underbrace{1}_{x'=1} \quad (**) \quad \left(\text{differentiate on both sides}\right)$$

Now: $\frac{(*)}{(**)} \Rightarrow \frac{\left(x^{\frac{p}{q}}\right)'}{1} = \frac{p\,u^{p-1} \cdot \cancel{u'}}{q\,u^{q-1} \cdot \cancel{u'}}$

$$\Rightarrow \left(x^{\frac{p}{q}}\right)' = \frac{p}{q}\,u^{p-1-(q-1)} = \frac{p}{q}\,u^{p-q} \underset{u = x^{\frac{1}{q}}}{=} \frac{p}{q}\,x^{\frac{p-q}{q}}$$

$$= \frac{p}{q}\,x^{\frac{p}{q}-1}$$

$r = \frac{p}{q}$

$$\Rightarrow \left(x^r\right)' = r\,x^{r-1} \quad , \; r \in \mathbb{Q}$$

This relation can be extended to arbitrary $r \in \mathbb{R}$ "by continuity" (i.e., r as a limit of rational numbers)

Example implicit differentiation:

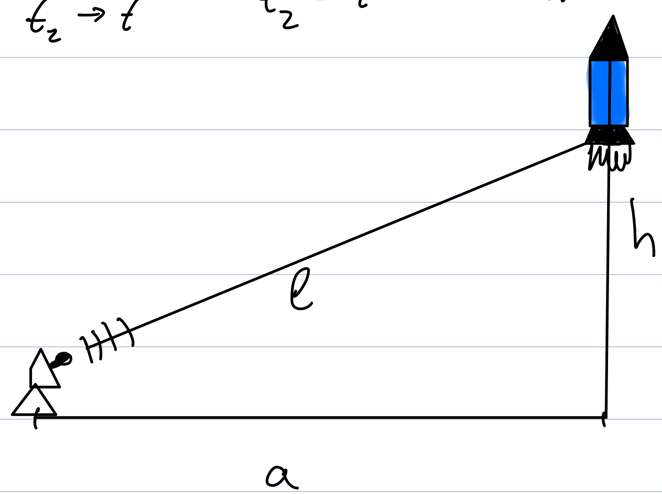
We have $x(t)$ as the distance traveled as a function of time t.

We then get: $\bar{v} = \dfrac{x(t_2) - x(t_1)}{t_2 - t_1} = \dfrac{\text{distance traveled between } t_1 \text{ and } t_2}{\text{travel time}}$

mean $\bar{\ }$ velocity v

$$v(t) = \lim_{t_2 \to t} \frac{x(t_2) - x(t)}{t_2 - t} = \frac{dx}{dt}$$ instantaneous velocity

<u>Now:</u>



Radar can track distance $l(t)$ and therefore approximate $l'(t) = \frac{dl}{dt}$

Q: What is the velocity (in the vertical) $\frac{dh}{dt}$ of the rocket?

$$l^2 = a^2 + h^2 \Rightarrow 2l\frac{dl}{dt} = 0 + 2h\frac{dh}{dt}$$

diff. w.r.t. t

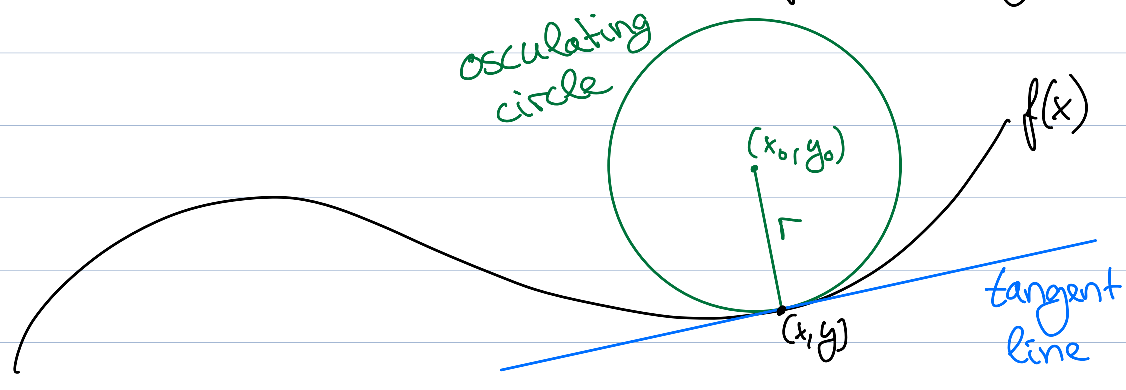
$$\Rightarrow \frac{dh}{dt} = \frac{l}{h}\frac{dl}{dt} = \frac{l}{\sqrt{l^2 - a^2}}\frac{dl}{dt},$$ $l, \frac{dl}{dt}, a$ are measurable quantities

Eg.: $a = 3\,km$, $\ell = 5\,km$, $\frac{d\ell}{dt} = 1000\,\frac{m}{s}$

$$\Rightarrow \frac{dh}{dt} = \underbrace{\frac{5\,km}{\sqrt{(25-9)\,km^2}}}_{=\frac{5}{4}\text{, units cancel}} \cdot 1000\,\frac{m}{s} = 1250\,\frac{m}{s}$$

Application of second derivative

Osculating circle:
- touches graph of f at point (x,y)
- it has the same first and second derivative as f at (x,y)



"osculating circle provides second order local approximation to graph of f, while tangent line is only first-order approximation."

Q: What is radius r of osculating circle?

① $(x-x_0)^2+(y-y_0)^2=r^2$ (general circle equation with centre (x_0,y_0) and radius r)

Differentiate in x:

② $2(x-x_0)\cdot 1+2(y-y_0)\frac{dy}{dx}=0$

$\Rightarrow\ x-x_0+(y-y_0)\frac{dy}{dx}$

Second derivative in x (diff. ②):

③ $$\underbrace{1+\frac{dy}{dx}\frac{dy}{dx}+(y-y_0)\frac{d^2y}{dx^2}}_{\text{product rule}}=0$$

Osculating circle conditions:

- $y=f(x)$
- $\frac{dy}{dx}=f'(x)$
- $\frac{d^2y}{dx^2}=f''(x)$

circle

graph

From 3: $(y-y_0) f''(x) = -1 - f'(x)^2$

$$\Rightarrow (y-y_0) = -\frac{1+f'(x)^2}{f''(x)}$$

From 2: $x - x_0 = -(y-y_0) f'(x)$

$$\overset{\text{use } 3}{=} -f'(x)\left(-\frac{1+f'(x)^2}{f''(x)}\right) = f'(x)\,\frac{1+f'(x)^2}{f''(x)}$$

From 1: $r^2 = (x-x_0)^2 + (y-y_0)^2$

$$\underset{2\,\&\,3}{\overset{\text{use}}{=}} \left(f'(x)\right)^2 \left(\frac{1+f'(x)^2}{f''(x)}\right)^2 + \left(\frac{1+f'(x)^2}{f''(x)}\right)^2$$

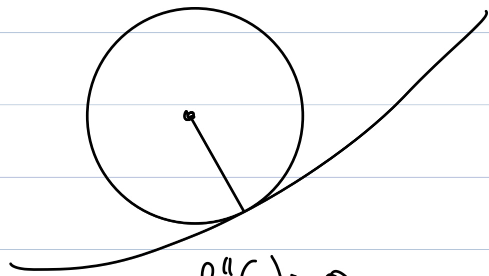
$$= \left(1+f'(x)^2\right)\left(\frac{1+f'(x)^2}{f''(x)}\right)^2$$

$$= \frac{\left(1+f'(x)^2\right)^3}{f''(x)^2}$$

Now define curvature

$$\boxed{K = \frac{1}{r} = \frac{\overbrace{f''(x)}^{\text{defines sign}}}{\underbrace{\left(1+f'(x)^2\right)^{\frac{3}{2}}}_{>0}}}$$

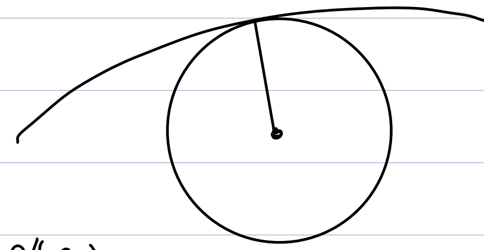
Link to concavity



$f''(x) > 0$

$\kappa > 0$

(circle above)



$f''(x) < 0$

$\kappa < 0$

(circle below)