Calculus and Elements of Linear Algebra I

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Live lectures sessions 11 & 12

# 2. Derivatives

## 2.1 Introduction to derivatives and their properties

and

## 2.2 Applications of differentiation

### Important consequence of MVT

$f : (a,b) \to \mathbb{R}$ diff'able with $f'(x) \geq 0$ ($\leq 0$)

$\Rightarrow$ $f$ is increasing (decreasing) on $(a,b)$

**Proof:** Suppose the contrary, i.e. there are $x_1 < x_2$ with $f(x_1) > f(x_2)$, $x_1, x_2 \in (a,b)$

Then: $< 0$

$\underline{MVT}$: $\underbrace{\dfrac{f(x_2)-f(x_1)}{\underbrace{(x_2-x_1)}_{>0}}}_{<0} = \underbrace{f'(c)}_{\geq 0}$ with $c \in (x_1, x_2)$

↯ contradiction

Opposite must be true!

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Example:

1. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$$f'(x) = 12x^3 - 12x^2 - 24x$$
$$= 12x(x^2 - x - 2)$$
$$= 12x(x+1)(x-2)$$

$\Rightarrow$ $f'(x) < 0$, so $f$ decreasing on $(-\infty, -1)$

$f'(x) > 0$, so $f$ increasing on $(-1, 0)$

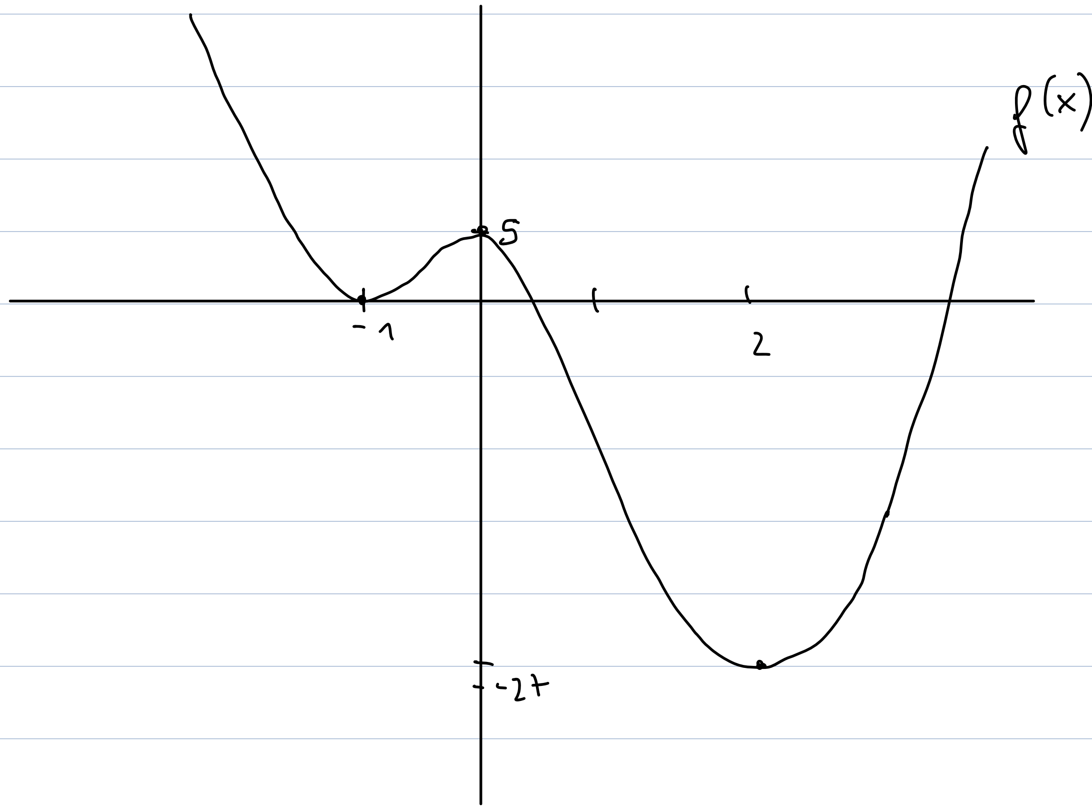
$f'(x) < 0$, so $f$ decreasing on $(0, 2)$

$f'(x) > 0$, so $f$ increasing on $(2, \infty)$

$\Rightarrow$ local min at $f(-1) = 0$ and $f(2) = -27$

local max at $f(0) = 5$

Can use this to sketch




② 

$$f(x) = \frac{x^2-1}{\sqrt{x^2+1}}$$

$$f'(x) = \frac{2x\left(\sqrt{x^2+1}\right) - (x^2-1)\cancel{\tfrac{1}{2}}(x^2+1)^{-\frac{1}{2}}\cdot \cancel{2}x}{x^2+1} \quad \frac{\cdot\left(\sqrt{x^2+1}\right)}{\cdot\left(\sqrt{x^2+1}\right)}$$

$$= \frac{2x(x^2+1) - (x^2-1)x}{(x^2+1)^{3/2}}$$

$$= \frac{2x^3+2x - x^3 + x}{(x^2+1)^{3/2}} = \frac{x\overbrace{(3+x^2)}^{>0}}{\underbrace{(x^2+1)^{3/2}}_{>0}}$$

$\Rightarrow f'(x) < 0$ if and only if $x < 0 \Rightarrow f$ decreasing

$f'(x) > 0$ if and only if $x < 0 \Rightarrow f$ increasing

$\Rightarrow f$ has a global minimum at $x = 0$
$(f(0) = -1)$

$$f''(x) = \frac{\left(1\cdot(3+x^2) + x\cdot 2x\right)(x^2+1)^{\cancel{3}\,\frac{3}{2}} - x\cdot(3+x^2)\cancel{2}x\cdot\frac{3}{\cancel{2}}\cancel{(x^2+1)^{\frac{1}{2}}}}{(x^2+1)^{\cancel{3}\,\frac{5}{2}}}$$

$$= \frac{3x^2+3 \cancel{+3x^4} + 3x^2 - 9x^2 \cancel{-3x^4}}{(x^2+1)^{5/2}}$$

$$= \frac{3-3x^2}{\underbrace{(x^2+1)^{5/2}}_{>0}}$$

Critical point at $x=0$: $f''(0) > 0$

$\Rightarrow f$ has a <u>local</u> min at $x=0$
(local because of the second derivative argument)
but we know from previous argument that it is actually global.

There is more information from $f''$:

$$f''(x) = 3\,\frac{1-x^2}{(x^2+1)^{5/2}} = 3\,\frac{(1-x)(1+x)}{\underbrace{(x^2+1)^{5/2}}_{>0}} \Rightarrow \text{roots } 1, -1$$

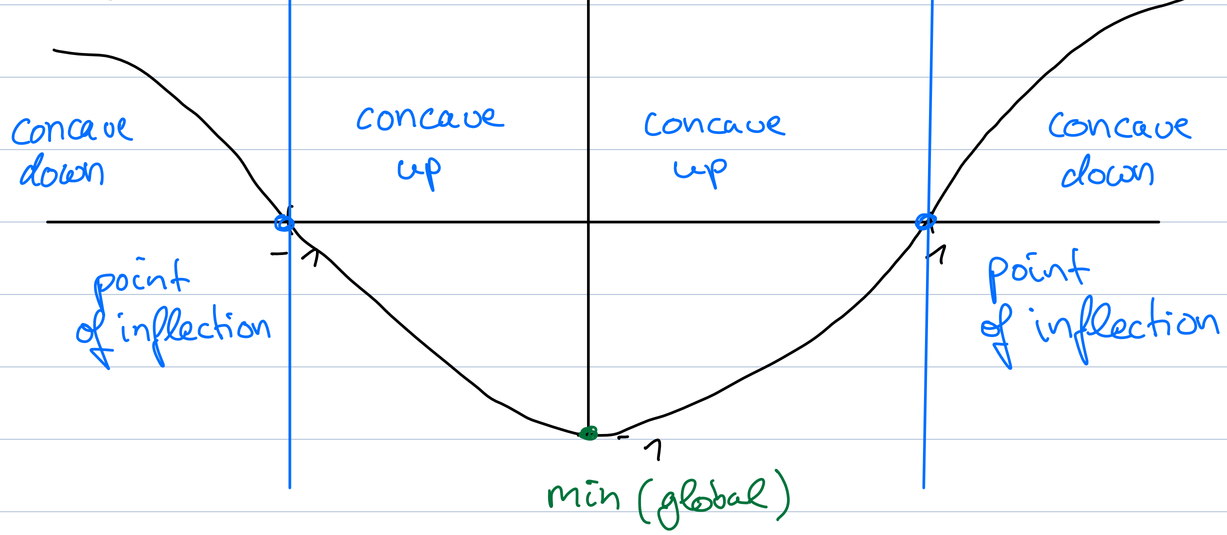
So at $x=-1$, $f''$ is changing sign from $-$ to $+$

$x=+1$, $f''$ is changing sign from $+$ to $-$

$\Rightarrow$ points of inflection at $x=-1$, $f(-1)=0$ and

$x=1$, $f(1)=0$

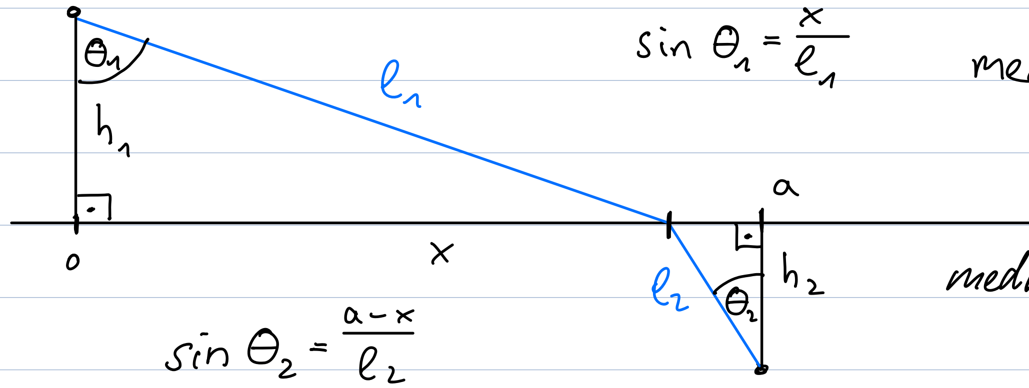
Sketch:




## Extreme value problem

Application of Snell's law:

Sketch:




$$\sin \theta_1 = \frac{x}{\ell_1}$$

$$\sin \theta_2 = \frac{a-x}{\ell_2}$$

$c_1$: speed of light in medium 1

$c_2$: speed of light in medium 2

Fermat's principle: light travels path of fastest propagation

i.e.: $T = \frac{\ell_1}{c_1} + \frac{\ell_2}{c_2}$ travel time takes minimum for the chosen path

($T$: total travel time light needs from source to observer)

We have: $h_1^2 + x^2 = \ell_1^2$ and $h^2 + (a-x)^2 = \ell_2^2$

Take derivative: $2x = 2\ell_1 \frac{d\ell_1}{dx} \Rightarrow \frac{d\ell_1}{dx} = \frac{x}{\ell_1} = \sin \theta_1$

$-2(a-x) = 2\ell_2 \frac{d\ell_2}{dx} \Rightarrow \frac{d\ell_2}{dx} = \frac{x-a}{\ell_2} = -\sin \theta_2$

(as $\ell_1$ and $\ell_2$ depend on $x$)

$$\Rightarrow \quad 0 \overset{!}{=} \frac{dT}{dx} = \frac{1}{c_1}\frac{dl_1}{dx} + \frac{1}{c_2}\frac{dl_2}{dx} \Rightarrow 0 = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}$$

$$\Rightarrow \boxed{\frac{c_1}{c_2} = \frac{\sin\theta_1}{\sin\theta_2} \qquad \text{Snell's law}}$$