Calculus and Elements of Linear Algebra I

Prof. Sören Petrat, Dr. Stephan Juricke (based on lecture notes by Marcel Oliver)

Jacobs University, Fall 2022

Live lectures sessions 13 & 14

2. Derivatives

2.2 Applications of differentiation

Another example for graph sketching:

$$f(x) = \sqrt[3]{\frac{x^2}{(x-6)^2}}$$

1. $D(f) = \mathbb{R} \setminus \{6\}$ domain

2. $f(0) = 0$, so $(0,0)$ is x- and y- intercept
   No further intercepts.

3. Hor. asymptotes:

$$\lim_{x\to\infty} f(x) = \sqrt[3]{\lim_{x\to\infty} \frac{x^2}{(x-6)^2}} = \sqrt[3]{\lim_{x\to\infty} \frac{1}{\Big(1-\underbrace{\tfrac{6}{x}}_{\xrightarrow[x\to\infty]{} 0}\Big)^2}} = 1$$

because $x \to \sqrt[3]{x}$ is cont.

$$= \lim_{x\to-\infty} f(x)$$

$\Rightarrow$ $y = 1$ is hor. asymptote for both $x \to \pm\infty$

4. Vertical asymptotes:

$$\lim_{x\to 6} f(x) = \infty \qquad \text{(both right and left sided limits)}$$

5. $$f(x) = \frac{x^{2/3}}{(x-6)^{2/3}}$$

$$f'(x) = \frac{\frac{2}{3}x^{-1/3}(x-6)^{2/3} - x^{2/3}\cdot\frac{2}{3}(x-6)^{-1/3}\cdot 1}{(x-6)^{4/3}}$$

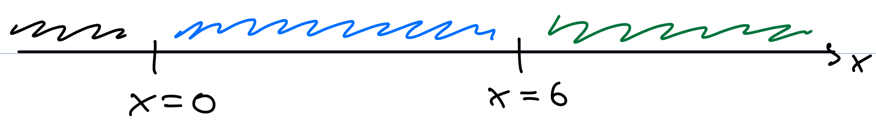
$$= \frac{2}{3}\, x^{-1/3}(x-6)^{-1/3}\, \frac{(\cancel{x}-6) - \cancel{x}}{(x-6)^{4/3}}$$

$$= -\frac{4}{x^{1/3}(x-6)^{5/3}}$$

$$D(f') = \mathbb{R} \setminus \{0, 6\}$$

To check sign of $f'(x)$, look at factors of $f'(x)$, i.e. $x^{-\frac{1}{3}}$ and $(x-6)^{-5/3}$ and when they change sign.

(Remark: you can take $\sqrt[3]{}$ of negative numbers and get real number!
e.g. $\sqrt[3]{(-1)\cdot(-1)\cdot(-1)} = -1 = \sqrt[3]{(-1)}$)



check signs on intervals

$x<0 \Rightarrow f'(x)<0 \quad \left(-4\cdot(<0)\cdot(<0) \Rightarrow \underbrace{-\cdot - \cdot -}_{=-}\right)$

f is decreasing

$x\in(0,6) \Rightarrow f'(x)>0$, f increasing

$x>6 \Rightarrow f'(x)<0$, f decreasing

$\Rightarrow$ at $x=0$, f has local min (note that $f'(0)$ does not exist)
$f(0)=0$

(the min. is actually global as hor. asymptotes are at $y=1>f(0)=0$ and vertical asymptote $\to +\infty$)

6.

$$f''(x) = -\frac{0\cdot x^{\frac{1}{3}}(x-6)^{\frac{5}{3}} - 4\cdot\left(\frac{1}{3}x^{-\frac{2}{3}}(x-6)^{\frac{5}{3}} + x^{\frac{1}{3}}\frac{5}{3}(x-6)^{\frac{2}{3}}\right)}{\left(x^{\frac{1}{3}}(x-6)^{\frac{5}{3}}\right)^2}$$

$$= +\frac{4}{3}x^{-\frac{2}{3}}(x-6)^{\frac{2}{3}}\;\frac{(x-6)+5x}{x^{\frac{2}{3}}(x-6)^{\frac{10}{3}}}$$

$$= \frac{4}{3}\;\frac{6(x-1)}{\underbrace{x^{4/3}}_{>0}\underbrace{(x-6)^{8/3}}_{>0}}$$

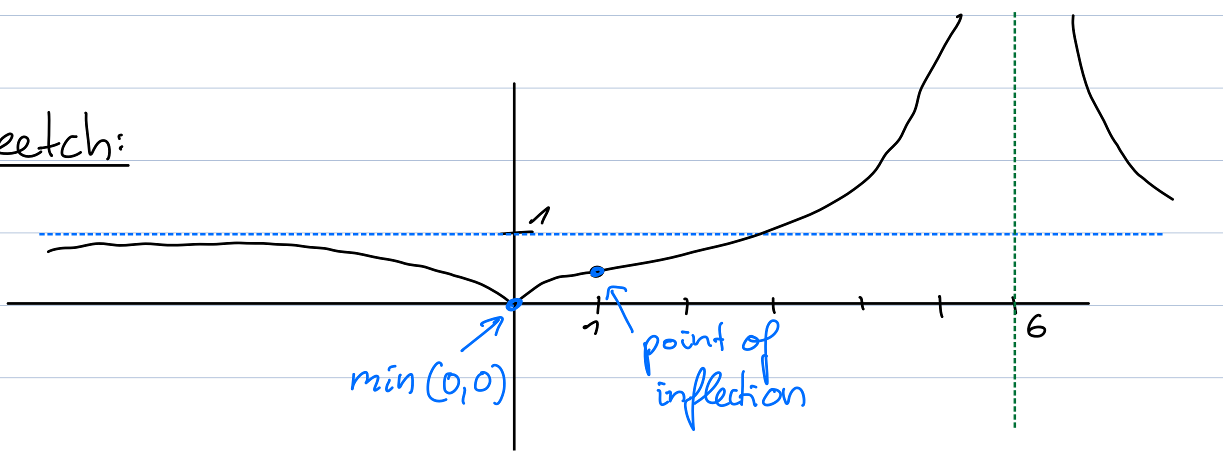
is responsible for sign

since numerator in exponent is even! e.g. $-2^{4/3} = \sqrt[3]{(-2)^4} > 0$

$\Rightarrow$ if $x < 1$, $f'' < 0$, so f concave (down)
$x > 1$, $f'' > 0$, so f concave up/convex

f has point of inflection at $x = 1$, because $f''(1) = 0$
and $f(1) = \frac{1}{(-5)^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{25}} \approx 0.34$

Sketch:



Calculus and Elements of Linear Algebra I — Example Session Week 7

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Example Session for:

Topic 2.2.B: Graph Sketching

Topic 3.A: Indefinite Integrals

Integration by Parts:

Example:
$$\int \underset{f'}{e^x}\,\underset{g}{\cos x}\,dx = \underset{f}{e^x}\,\underset{g}{\cos x} - \int \underset{f}{e^x}\,\underset{g'}{(-\sin x)}\,dx$$

$$= e^x \cos x + \underbrace{\int e^x \sin x\,dx}_{\overset{\text{int. by parts again}}{=}\; e^x \sin x - \int e^x \cos x\,dx}$$

$$\Rightarrow \int e^x \cos x\,dx = e^x(\cos x + \sin x) - \int e^x \cos x\,dx + c$$

$$\Rightarrow \int e^x \cos x\,dx = \frac{1}{2} e^x (\cos x + \sin x) + c\,.$$

Example:
$$\int \ln x\,dx = \int \overset{f'}{1}\cdot \overset{g}{\ln x}\,dx = \overset{f}{x}\,\overset{g}{\ln x} - \int \overset{f}{x}\,\overset{g'}{\frac{1}{x}}\,dx = x \ln x - x + c\,.$$

Integration by Substitution:

Example: $\int \tan x \, dx = \int \frac{1}{\cos x} \underbrace{\sin x \, dx}_{= u'(x)dx} \Rightarrow u(x) = -\cos x \ , f(y) = \frac{1}{y}$

$$= -\int \frac{1}{u} du = -\ln|u| + c \ , u \neq 0$$

$$= -\ln|\cos(x)| + c \quad \text{(for intervals not including the zeroes of cos).}$$