Calculus and Elements of Linear Algebra I Example Session Week 8

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Example Session for:

Topic 3.B: Integration of Rational Functions

Topic 3.C: Definite Integrals and the Fundamental Theorem of Calculus

Integrals of Rational Functions:

Also works for polynomials of trigonometric functions.

Example: $\int \frac{1}{\cos x}\,dx$. Trick: set $u(x) = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow$ "$dx = \frac{1}{\cos x}\,du$"

$$\Rightarrow \int \frac{1}{\cos x}\,dx = \int \frac{1}{\cos^2 x}\,du \quad \text{and} \quad \cos^2 x + u^2(x) = 1, \text{ i.e., } \cos^2 x = 1 - u^2(x)$$

$$= \int \frac{1}{1-u^2}\,du$$

partial fractions: $\frac{1}{1-u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u} = \frac{A(1+u) + B(1-u)}{(1-u)(1+u)}$

$$\Rightarrow 1 = (A-B)u + (A+B) \Rightarrow A = \tfrac{1}{2} = B.$$

$$\Rightarrow \int \frac{1}{1-u^2}\,du = \frac{1}{2}\underbrace{\int \frac{1}{1-u}\,du}_{= -\int \frac{1}{u-1}du} + \frac{1}{2}\int \frac{1}{1+u}\,du = \frac{1}{2}\left(-\ln|u-1| + \ln|u+1|\right) + c$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + c$$

$$\Rightarrow \int \frac{1}{\cos x}\,dx = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + c.$$

Definite Integral "by hand":

$$\int_0^1 e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{x_{i-1}}\,\Delta x \qquad \text{, here } \Delta x = \frac{1}{n} \text{ , } x_i = 0 + i\Delta x = \frac{i}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} e^{\frac{i-1}{n}}\,\frac{1}{n}$$

$$= \lim_{n\to\infty} \sum_{i=0}^{n-1} e^{\frac{i}{n}}\,\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(e^{\frac{1}{n}}\right)^i$$

Note: $\sum_{i=0}^{n-1} q^i$ is called a $\boxed{\text{"geometric sum"}}$.

Let $S_n := \sum_{i=0}^{n-1} q^i = 1 + q + q^2 + \dots + q^{n-1}$.

Then $qS_n = \sum_{i=0}^{n-1} q^{i+1} = q + q^2 + \dots + q^{n-1} + q^n = \sum_{i=1}^{n} q^i$

$$\Rightarrow \underbrace{S_n - qS_n}_{= S_n(1-q)} = \sum_{i=0}^{n-1} q^i - \sum_{i=1}^{n} q^i = 1 - q^n$$

$$\Rightarrow S_n = \frac{1-q^n}{1-q}.$$

We continue: $\int_0^1 e^x\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left(e^{\frac{1}{n}}\right)^i$

$$= \lim_{n\to\infty} \frac{1}{n}\,\frac{1-\left(e^{\frac{1}{n}}\right)^n}{1-e^{\frac{1}{n}}}$$

$$= (1-e)\,\underbrace{\lim_{n\to\infty} \frac{1}{n}\,\frac{1}{1-e^{\frac{1}{n}}}}_{= \lim_{h\to\infty} \frac{h}{1-e^h} = \frac{1}{\lim_{h\to 0} \frac{1-e^h}{h}} = \frac{1}{-\exp'(0)} = \frac{-1}{e^0} = -1}$$

$$= e - 1$$

Integral Mean-Value Theorem:

If $f:[a,b]\to\mathbb{R}$ is continuous, then there exists $z\in[a,b]$ s.t. $\frac{1}{b-a}\int_a^b f(x)dx = f(z)$.

Proof:

Let $m := \min_{x\in[a,b]} f(x)$

$M := \max_{x\in[a,b]} f(x)$

the limits exist bc. f is cont. on a closed bounded interval (extreme value thm.)

$\Rightarrow m \le f(x) \le M$ (*)

$\Rightarrow \underbrace{\int_a^b m\,dx}_{=(b-a)m} \le \int_a^b f(x)dx \le \underbrace{\int_a^b M\,dx}_{=(b-a)M}$ by Theorem (ii), Session 16.

$\Rightarrow m \le \frac{1}{(b-a)}\int_a^b f(x)dx \le M$

From (*) and the intermediate value thm. we know that f assumes every value between m and M, so in particular there is a $z\in[a,b]$ s.t. $f(z)=\frac{1}{b-a}\int_a^b f(x)dx$.

$\square$