Calculus and Elements of Linear Algebra I — Example Session Week 9

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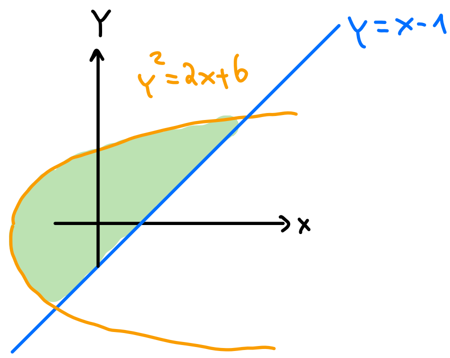
Example Session for:

Topic 3.D: Applications of Integration

Topic 3.E: Improper Integrals

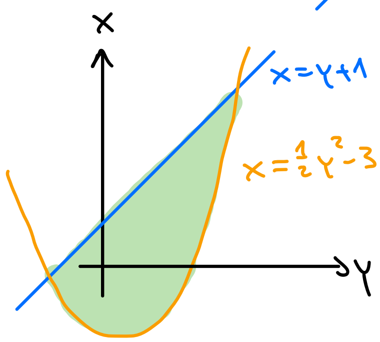
Area between Curves:

How about area between $\underbrace{y^2 = 2x+6}_{y \text{ is not a fct. of } x \text{ here}}$ and $y = x-1$?



$\Rightarrow$ Express x as fct. of y!

$\Rightarrow$ $x = \frac{1}{2}y^2 - 3$ and $x = y+1$:



Points of intersection: $\frac{1}{2}y^2 - 3 = y+1 \Rightarrow y^2 - 2y - 8 = 0$

$$\Rightarrow y = 1 \pm \sqrt{1+8} \Rightarrow y = -2 =: a \text{ and } y = 4 =: b$$

$$\Rightarrow \text{area } A = \int_{-2}^{4} \left(y+1-\left(\tfrac{1}{2}y^2-3\right)\right)dy = \int_{-2}^{4}\left(-\tfrac{1}{2}y^2 + y + 4\right)dy$$

$$= \left[-\tfrac{1}{6}y^3 + \tfrac{1}{2}y^2 + 4y\right]_{-2}^{4} = -\tfrac{1}{6}\cdot 64 + \tfrac{1}{2}\cdot 16 + 16 - \left(\tfrac{1}{6}\cdot 8 + \tfrac{1}{2}\cdot 4 - 8\right)$$

$$= \ldots = 18.$$

Gamma Function:

The gamma fct. is defined as $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx$.

We have: • $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = -(0 - e^0) = 1.$

• $\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$

$$\overset{\text{int. by parts}}{=} \underbrace{-e^{-x}x^n\Big|_0^\infty}_{=\lim\limits_{r\to\infty}\left(-\frac{r^n}{e^r}+0\right)} - \int_0^\infty (-e^{-x})\, n x^{n-1}\, dx$$

$= 0$ (exp grows faster than any polynomial!)

$$\Rightarrow \Gamma(n+1) = n\int_0^\infty x^{n-1} e^{-x}\, dx = n\,\Gamma(n).$$

Thus, $\Gamma(n) = (n-1)(n-2)\cdots 1 = (n-1)!$ "$(n-1)$ factorial"

$\Gamma(x)$ is a generalization of the factorial to non-integer values (shifted by 1).

Taylor Series:

Goal: approximate $f(x)$ near b by a power series $\sum_{k=0}^{N} \frac{(x-b)^k}{k!} c_{k,b}$

By the FTC, we know: $\int_b^x f'(t)\,dt = f(x) - f(b)$

$$\Rightarrow f(x) = f(b) + \int_b^x f'(t)\,dt$$

Integration by parts: $\int_b^x \underbrace{1}_{=\frac{d}{dt}(t-x)} \cdot f'(t)\,dt = (t-x) f'(t)\Big|_b^x - \int_b^x (t-x) f''(t)\,dt$

$$= (x-b) f'(b) + \int_b^x (x-t) f''(t)\,dt$$

$$\Rightarrow f(x) = \underbrace{f(b) + (x-b) f'(b)}_{\text{linear approximation}} + \int_b^x (x-y) f''(y)\,dy$$

Another integration by parts yields (check the computation!):

$$f(x) = \underbrace{f(b) + (x-b) f'(b) + \frac{(x-b)^2}{2} f''(b)}_{\text{second order approximation (parabola)}} + \int_b^x \frac{(x-y)^2}{2} f'''(y)\,dy$$

Repeating this yields:

Theorem (Taylor expansion):

Let $f: \mathbb{R} \to \mathbb{R}$ be $(N+1)$ times continuously differentiable on $[b, x]$. Then

$$f(x) = \sum_{k=0}^{N} \frac{(x-b)^k}{k!} f^{(k)}(b) + \underbrace{\int_b^x \frac{(x-y)^N}{N!} f^{(N+1)}(y)\,dy}_{=: R_N(x),\ \text{called remainder}}.$$

Note: The Taylor series is useful if $R_N(x)$ goes to 0 for x sufficiently close to b. In that case $f(x) = \sum_{k=0}^{\infty} \frac{(x-b)^k}{k!} f^{(k)}(b) := \lim_{N\to\infty} \sum_{k=0}^{N} \frac{(x-b)^k}{k!} f^{(k)}(b)$.

Needs to be checked for examples.