Week 10: ODEs

1. MULTI Single

Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = y + 1$ with initial condition $y(0) = 1$.

(a) $y(t) = 1$
(b) $y(t) = 3e^t - 1$
(c) $y(t) = 3e^t - 2$
(d) $y(t) = 2e^t - 1$

2. MULTI Single

Solve $\frac{\mathrm{d}y}{\mathrm{d}t} = -2yt^2$ with initial condition $y(0) = 2$.

(a) $y(t) = 3e^{-t^3} - 1$
(b) $y(t) = 2e^{-t^3}$
(c) $y(t) = e^{-\frac{t^3}{3}} + 1$
(d) $y(t) = 2e^{-\frac{2t^3}{3}}$

3. MULTI Single

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} - 3e^x = ye^x$. (*Note:* A, C are constants in the answers below.)

(a) $y = 3^{e^x} - 3 + C$
(b) $y = Ae^{e^x} - 3$
(c) $y = e^{e^x} - 3 + C$
(d) $y = Ae^{3e^x}$

4. MULTI Single

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y + x - 1$. (*Note:* A, C are constants in the answers below.)

(a) $y = Ae^x - x - 1$
(b) $y = e^x - x$
(c) $y = Ae^x - x$
(d) $y = e^x - x + C$

5. MULTI Single

For each of the following equations, determine all the equilibrium points (where $y'(x) = 0$) and classify each as stable (y' changes sign from positive to negative at x or unstable (y' changes sign from negative to positive at x).

- $y_1' = y_1 - y_1^2$
- $y_2' = y_2(y_2 - 1)(y_2 - 2)$
- $y_3' = e^{y_3} - 1$

(a) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (stable)

(b) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (unstable)
(c) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1$ (unstable), $y_2 = 0, 2$ (stable), $y_3 = 0$ (stable)
(d) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1, 2$ (unstable), $y_2 = 0$ (stable), $y_3 = 0$ (stable)

6. MULTI Single

What are the transversal oscillation frequencies ω of a string whose endpoints are fixed ($y(t, x = 0) = y(t, x = a) = 0$, see the definition of y below)? Note that by denoting deviation of a string from $y = 0$ along $x = \tilde{x}$ at time t by $y(t, \tilde{x})$, and using Newton's II law with the small oscillations approximation as well as assuming that all the points on a string oscillate in phase, one obtains the following equation of motion for the spatial part $y(x)$ of $y(t, x) = y(x) \sin(\omega t + \varphi)$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\alpha\omega^2 y$$

(a) $\omega = \sqrt{\alpha}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$
(b) $\omega = \frac{1}{\sqrt{\alpha}}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$
(c) $\omega = \frac{1}{\sqrt{\alpha}}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$
(d) $\omega = \sqrt{\alpha}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$

7. MULTI Single

Find the solution to the equation $\hat{a}\psi(x) = 0$, where the action of the operator $\hat{a}$ is given by $\hat{a} := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and $\psi(x)$ is a function of x.
Remark: This is the ground-state solution to the Quantum Harmonic Oscillator in physics.

(a) $A \tanh x$
(b) $A \cos(b \cdot x)$
(c) Ae^{-x}
(d) $Ae^{-\frac{x^2}{2}}$

8. MULTI Single

Consider a fly whose x coordinate changes over time with the equation of motion given by $\dot{x} = \sin(x)$, where $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$.
If its initial position is given by $x_0 = \pi/4$, what will happen to the fly?
Hint: Integrating $\frac{1}{\sin x}$ is not necessary for finding the solution.

(a) It will steadily move towards the right until it reaches π
(b) It will diverge towards infinity

(c) It will accelerate towards the right until it reaches $\frac{\pi}{2}$ and then slow down until finally reaching π
(d) Its position will oscillate indeterminately

9. MULTI Single

The rate of change of the volume of a spherical snowball that is melting is proportional to its area. What is the equation describing the radius of the snowball as a function of time?

(a) $r(t) = r_0 \cdot e^{-ct}$
(b) $r(t) = r_0 - \frac{5}{2}(ct)^{\frac{5}{2}}$
(c) $r(t) = r_0 - ct$
(d) $r(t) = r_0 - \sqrt{ct}$

10. MULTI Single

What is the angle (in radian, i.e., where 360° corresponds to 2π) between the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix}?$$

(a) 0
(b) π
(c) $\frac{\pi}{2}$
(d) $\frac{\pi}{4}$

Total of marks: 10