Week 10: ODEs

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} = y + 1$ with initial condition $y(0) = 1$.

(a) $y(t) = 2e^t - 1$ (100%)
(b) $y(t) = 3e^t - 1$
(c) $y(t) = 1$
(d) $y(t) = 3e^t - 2$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y + 1 \Rightarrow y + 1 = Ae^t$$

and using $y(0) = 1$ one obtains:

$$y(t) = 2e^t - 1$$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -2yt^2$ with initial condition $y(0) = 2$.

(a) $y(t) = 2e^{-t^3}$
(b) $y(t) = 2e^{-\frac{2t^3}{3}}$ (100%)
(c) $y(t) = 3e^{-t^3} - 1$
(d) $y(t) = e^{-\frac{t^3}{3}} + 1$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -2yt^2 \Rightarrow \frac{\mathrm{d}y}{y} = -2t^2\,\mathrm{d}t \Rightarrow y(t) = Ae^{-\frac{2t^3}{3}}$$

using $y(0) = 2$ one obtains:

$$y(t) = 2e^{-\frac{2t^3}{3}}$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3e^x = ye^x$. (*Note:* A, C are constants in the answers below.)

(a) $y = Ae^{e^x} - 3$ (100%)
(b) $y = e^{e^x} - 3 + C$
(c) $y = 3^{e^x} - 3 + C$
(d) $y = Ae^{3e^x}$

Rearrange $\dfrac{\mathrm{d}y}{\mathrm{d}x} - 3e^x = ye^x$ *into* $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (y+3)e^x \Rightarrow \dfrac{\mathrm{d}y}{y+3} = e^x\mathrm{d}x$ *to see that the equation is separable. Integrate to get:* $\ln(y+3) = e^x + \tilde{A} \Rightarrow y = e^{e^x + \tilde{A}} - 3 = Ae^{e^x} - 3$

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = y + x - 1$. (*Note:* A, C are constants in the answers below.)

(a) $y = Ae^x - x$ (100%)
(b) $y = e^x - x + C$
(c) $y = Ae^x - x - 1$
(d) $y = e^x - x$

Define $z := y + x$, *then noting that* $\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x} + 1$ *the equation becomes:* $\frac{\mathrm{d}z}{\mathrm{d}x} = z$ *which is a separable equation with the following solution:* $z = Ae^x \Rightarrow y = Ae^x - x$

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

For each of the following equations, determine all the equilibrium points (where $y'(x) = 0$) and classify each as stable (y' changes sign from positive to negative at x or unstable (y' changes sign from negative to positive at x).

- $y_1' = y_1 - y_1^2$
- $y_2' = y_2(y_2 - 1)(y_2 - 2)$
- $y_3' = e^{y_3} - 1$

(a) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (unstable) (100%)
(b) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1$ (unstable), $y_2 = 0, 2$ (stable), $y_3 = 0$ (stable)
(c) $y_1 = 0$ (unstable), $y_1 = 1$ (stable), $y_2 = 0, 2$ (unstable), $y_2 = 1$ (stable), $y_3 = 0$ (stable)
(d) $y_1 = 1$ (unstable), $y_1 = 0$ (stable), $y_2 = 1, 2$ (unstable), $y_2 = 0$ (stable), $y_3 = 0$ (stable)

$$y_1' = y_1(1-y_1) \begin{cases} = 0 & \textit{for } y_1 = 0, 1 \\ < 0 & \textit{for } y_1 < 0, y_1 > 1 \\ > 0 & \textit{for } y_1 \in (0,1) \end{cases}$$

Thus, the equilibrium points are: $y_1 = 0$ *(unstable),* $y_1 = 1$ *(stable)*

$$y_2' = y_2(y_2-1)(y_2-2) \begin{cases} = 0 & \textit{for } y_2 = 0, 1, 2 \\ < 0 & \textit{for } y_2 < 0, y_2 \in (1,2) \\ > 0 & \textit{for } y_2 \in (0,1), y_2 > 0 \end{cases}$$

Thus, the equilibrium points are: $y_2 = 0, 2$ *(unstable),* $y_2 = 1$ *(stable)*

$$y_3' = e^{y_3} - 1 \begin{cases} = 0 & \textit{for } y_3 = 0 \\ < 0 & \textit{for } y_3 < 0 \\ > 0 & \textit{for } y_3 > 0 \end{cases}$$

Thus, the equilibrium point is: $y_3 = 0$ *(unstable)*

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What are the transversal oscillation frequencies ω of a string whose endpoints are fixed ($y(t, x=0) = y(t, x=a) = 0$, see the definition of y below)? Note that by denoting deviation of a string from $y = 0$ along $x = \tilde{x}$ at time t by $y(t, \tilde{x})$, and using Newton's II law with the small oscillations approximation as well as assuming that all the points on a string oscillate in phase, one obtains the following equation of motion for the spatial part $y(x)$ of $y(t,x) = y(x)\sin(\omega t + \varphi)$:

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\alpha\omega^2 y$$

(a) $\omega = \frac{1}{\sqrt{\alpha}}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N} = \{1, 2, 3, \dots\}$ (100%)

(b) $\omega = \sqrt{\alpha}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N} = \{1, 2, 3, \dots\}$

(c) $\omega = \frac{1}{\sqrt{\alpha}}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

(d) $\omega = \sqrt{\alpha}\left[\frac{\pi k}{a}\right]$ where $k \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$

General solution to $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\alpha\omega^2 y$ *is given by* $y(x) = A\sin(\omega\sqrt{\alpha}\,x) + B\cos(\omega\sqrt{\alpha}\,x)$. *Using boundary conditions:*

$$y(x=0) = 0 + B = 0 \Rightarrow B = 0, y(x=a) = A\sin(a\omega\sqrt{\alpha}) = 0 \Rightarrow a\omega\sqrt{\alpha} = \pi k, k \in \mathbb{Z}$$

Using the fact that the frequency of oscillations is positive ($\omega = 0$ *corresponds to no oscillations) one concludes:* $\omega = \frac{1}{\sqrt{\alpha}}\left[\frac{\pi k}{a}\right]$ *where* $k \in \mathbb{N} = \{1, 2, 3, \dots\}$

7.

MULTI 1.0 point 0 penalty Single Shuffle

Find the solution to the equation $\hat{a}\psi(x) = 0$, where the action of the operator $\hat{a}$ is given by $\hat{a} := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$ and $\psi(x)$ is a function of x.
Remark: This is the ground-state solution to the Quantum Harmonic Oscillator in physics.

(a) $Ae^{-\frac{x^2}{2}}$ (100%)
(b) Ae^{-x}
(c) $A\cos(b \cdot x)$
(d) $A\tanh x$

$$\hat{a}\psi = \frac{x}{\sqrt{2}}\psi + \frac{1}{\sqrt{2}}\frac{\mathrm{d}\psi}{\mathrm{d}x} = 0 \implies \frac{\mathrm{d}\psi}{\mathrm{d}x} = -x \cdot \psi$$

This is separable ODE: $\frac{\mathrm{d}\psi}{\psi} = -\frac{\mathrm{d}x}{x} \implies \ln\psi = -\frac{x^2}{2} + C \implies \psi(x) = Ae^{-\frac{x^2}{2}}$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Consider a fly whose x coordinate changes over time with the equation of motion given by $\dot{x} = \sin(x)$, where $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$.
If its initial position is given by $x_0 = \pi/4$, what will happen to the fly?
Hint: Integrating $\frac{1}{\sin x}$ is not necessary for finding the solution.

(a) It will accelerate towards the right until it reaches $\frac{\pi}{2}$ and then slow down until finally reaching π (100%)
(b) It will steadily move towards the right until it reaches π
(c) It will diverge towards infinity
(d) Its position will oscillate indeterminately

The rate of change of the x coordinate is given by $\sin(x)$. *This function is positive at $x = \frac{\pi}{4}$ and thus the x coordinate increases initially (moves towards the right). The function reaches a maximum at $\frac{\pi}{2}$, where the fly will have maximum velocity. $\dot{x}$ is positive until reaching $x = \pi$, where $\dot{x} = 0$, meaning it has no velocity in the x direction (and thus stops moving).*

9.

MULTI 1.0 point 0 penalty Single Shuffle

The rate of change of the volume of a spherical snowball that is melting is proportional to its area. What is the equation describing the radius of the snowball as a function of time?

(a) $r(t) = r_0 - ct$ (100%)
(b) $r(t) = r_0 - \sqrt{ct}$

(c) $r(t) = r_0 - \frac{5}{2}(ct)^{\frac{5}{2}}$
(d) $r(t) = r_0 \cdot e^{-ct}$

$$\frac{\mathrm{d}}{\mathrm{d}t}V(r) \propto A(r) \implies \frac{\mathrm{d}V}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = -c \cdot A \qquad \textit{for some } c \in \mathbb{R}^+$$
$$\dot{r}\frac{\mathrm{d}}{\mathrm{d}r}\frac{4\pi}{3}r^3 = -c \cdot 4\pi r^2 \implies \dot{r} = -c \implies r(t) = r_0 - ct$$

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What is the angle (in radian, i.e., where 360° corresponds to 2π) between the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

and

$$\begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix}?$$

(a) 0
(b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (100%)
(d) π

Recall the formula for the scalar product: $u \cdot v = |u||v|\cos(\theta)$, with θ the angle between the vectors u and v. Here, we find

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix} = 0,$$

i.e., the angle is $\frac{\pi}{2}$ (or 90°).

Total of marks: 10