Week 11: Vectors

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

A line is given by $\vec{r} = \lambda\vec{a} + \vec{b}$, with $\vec{a} = (1, -1, 4)^T$ and $\vec{b} = (4, 5, 6)^T$, while the equation of a plane is given by $-2x + 2y + z = 17$. What are the coordinates of the point P where the line and plane intersect?

(a) The line and the plane do not intersect (100%)
(b) $P = (-1, 4, 7)$
(c) $P = (3, 3, 17)$
(d) The line and the plane intersect infinitely many times

Consider the vector normal to the plane $\vec{n} = (-2, 2, 1)^T$.
One can observe $\vec{a} \cdot \vec{n} = 1 \cdot (-2) + (-1) \cdot 2 + 4 \cdot 1 = 0$. Thus, the line and the plane either do not intersect or intersect infinitely many times.
Analyzing $\lambda = 0$ and plugging in the coordinates of the line into the plane equation:
$-2 \cdot 4 + 2 \cdot 5 + 1 \cdot 6 = 8 \neq 17$.
Thus, the line and plane do not intersect.

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

What is the equation of the hyperplane, given by $\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \vec{p}_0 + \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$ with

$$\vec{p}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \vec{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \alpha, \beta, \gamma \in \mathbb{R}$$

(a) $-t - x - y + z + 1 = 0$ (100%)
(b) $-t - x - y - z + 1 = 0$
(c) $t + x - y + z - 1 = 0$
(d) $t + x - y - z - 1 = 0$

Note that $\vec{d} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ +1 \end{bmatrix}$ *is perpendicular to* $\vec{a}, \vec{b}, \vec{c}$.

This defines the following equation (expanded version of $\vec{d} \cdot \left(\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} - \vec{p}_0 \right) = 0$*):*

$$-(t-1) - x - y + z = 0 \Leftrightarrow -t - x - y + z + 1 = 0$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the cross product $\vec{u} \times \vec{v}$ of $\vec{u} = \langle 3, 2, -1 \rangle$, $\vec{v} = \langle 1, 1, 0 \rangle$

(a) $\langle 1, -1, 1 \rangle$ (100%)
(b) $\langle -1, -1, 5 \rangle$
(c) $\langle -6, 4, 2 \rangle$
(d) $\langle 6, 4, 2 \rangle$

Direct computation

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the unit vector along the direction of the cross product $\vec{u} \times \vec{v}$ of $\vec{u} = \langle 7, -1, 3 \rangle$, $\vec{v} = \langle 2, 0, -2 \rangle$.

(a) $\frac{1}{\sqrt{408}} \langle 2, 20, 2 \rangle$ (100%)
(b) $\frac{1}{108} \langle -2, -10, 2 \rangle$
(c) $\frac{1}{408} \langle 2, 20, 2 \rangle$
(d) $\frac{1}{\sqrt{108}} \langle -2, -10, 2 \rangle$

Direct computation yields:

$$\vec{u} \times \vec{v} = \begin{bmatrix} 2 \\ 20 \\ 2 \end{bmatrix}$$

Then the norm squared is simply $2^2 + 20^2 + 2^2 = 408$. *Thus, we take the square root to find the norm and divide by it to normalize.*

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i\,j\,k) = (1\,2\,3),\ (2\,3\,1),\ \text{or } (3\,1\,2) \\ -1 & \text{if } (i\,j\,k) = (1\,3\,2),\ (3\,2\,1),\ \text{or } (2\,1\,3) \\ 0 & \text{else} \end{cases}$

Consider $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$. Which of the following is equivalent to the kth component of $\vec{u} \times \vec{v}$

(a) $[\vec{u} \times \vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} u_i v_j$ (100%)

(b) $[\vec{u} \times \vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} v_i u_j$

(c) $[\vec{u} \times \vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk} (u_i v_j - v_i u_j)$

(d) $[\vec{u}\times\vec{v}]_k = \sum_{i,j=1}^{3} \epsilon_{ijk}(u_i\, v_j + v_i\, u_j)$

Direct computation yields:

$$\vec{u}\times\vec{v} = \begin{bmatrix} u_2\,v_3 - u_3\,v_2 \\ u_3\,v_1 - u_1\,v_3 \\ u_1\,v_2 - u_2\,v_1 \end{bmatrix} = \begin{bmatrix} \epsilon_{231}u_2\,v_3 + \epsilon_{321}u_3\,v_2 + 0 \\ \epsilon_{312}u_3\,v_1 + \epsilon_{132}u_1\,v_3 + 0 \\ \epsilon_{123}u_1\,v_2 + \epsilon_{213}u_2\,v_1 + 0 \end{bmatrix} = \sum_{i,j=1}^{3} \begin{bmatrix} \epsilon_{ij1}u_i\,v_j \\ \epsilon_{ij2}u_i\,v_j \\ \epsilon_{ij3}u_i\,v_j \end{bmatrix}$$

Where the $+0$ represents all the other ϵ_{ijk} terms.

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find a basis for $\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| 7x + 2y - 5z = 0 \right\} \subset \mathbb{R}^3$.

(a) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$ (100%)

(b) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

(c) $\begin{bmatrix} 5 \\ 0 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 10 \\ 5 \\ 14 \end{bmatrix}$

The plane is spanned by vectors, perpendicular to the normal vector $\vec{n} = \begin{bmatrix} 7 \\ 2 \\ -5 \end{bmatrix}$

For a basis take a vector, perpendicular to $\vec{n}$, e.g. $\vec{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$, *and add another one which is not proportional to the first one (to have 2 linearly independent vectors which form a basis of a plane), e.g.* $\vec{v}_2 = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find a basis for $\left\{ \begin{bmatrix} 3a \\ -7a \\ 11a \end{bmatrix} \in \mathbb{R}^3 \middle| a \in \mathbb{R} \right\} \subset \mathbb{R}^3$.

(a) $\begin{bmatrix} 15 \\ -35 \\ 55 \end{bmatrix}$ (100%)
(b) $\begin{bmatrix} 51 \\ -118 \\ 187 \end{bmatrix}$
(c) $\begin{bmatrix} -3 \\ -7 \\ 11 \end{bmatrix}$
(d) $\begin{bmatrix} 4 \\ 7 \\ 4 \end{bmatrix}$

A line is spanned by one vector (such that any point on the line is proportional to the basis vector), thus we can take e.g. $\begin{bmatrix} 3 \\ -7 \\ 11 \end{bmatrix}$ *to be a basis vector, or any other multiple of it, e.g.* $\begin{bmatrix} 3 \\ -7 \\ 11 \end{bmatrix} \cdot 5 = \begin{bmatrix} 15 \\ -35 \\ 55 \end{bmatrix}$

8.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is not a basis for the space of all cubic polynomials $P_3(\mathbb{R})$?

(a) $\mathfrak{B} = \{x^3 - x^2,\ x^3 - x,\ x^2 - x,\ x^3 - 1\}$ (100%)
(b) $\mathfrak{B} = \{x^3,\ x^2,\ x,\ 1\}$
(c) $\mathfrak{B} = \{x^3 - x^2,\ x^2 - x,\ x - 1,\ 1\}$
(d) $\mathfrak{B} = \{x^3 + x^2 + x + 1,\ (x - 6)^2,\ x - 10,\ 1\}$

We can see: $(-1)(x^3-x^2)+(1)(x^3-x)+(-1)(x^2-x) = -x^3+x^2+x^3-x-x^2+x = 0$
This means that three of the basis vectors are linearly dependent, and thus cannot be a basis.

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Does the set of all positive reals together with the following addition and multiplication by scalar $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ form a vector space over $\mathbb{R}$ (with the scalars $c \in \mathbb{R}$):

$$v_1 \tilde{+} v_2 \overset{def}{=} v_1 \cdot v_2; \quad c \tilde{\cdot} v_2 \overset{def}{=} c \cdot v_2$$

(a) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is not a vector space over $\mathbb{R}$ (100%)
(b) $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ is a vector space over $\mathbb{R}$

The space is not closed under multiplication by a scalar, note e.g. that:

$$-1 \tilde{\cdot} 1 = -1 \notin \mathbb{R}_+$$

Thus, $(\mathbb{R}_+, \tilde{+}, \tilde{\cdot})$ *is not a vector space over* $\mathbb{R}$

10.

MULTI 1.0 point 0 penalty Single Shuffle

Is the set of all polynomials in one variable with real coefficients of degree 10 a vector space over $\mathbb{R}$ (let's denote this space by $\mathbb{R}[X]^{10}$)? (Addition and multiplication by scalar are defined as usual).

(a) $\mathbb{R}[X]^{10}$ is not a vector space over $\mathbb{R}$ (100%)
(b) $\mathbb{R}[X]^{10}$ is a vector space over $\mathbb{R}$

Elements of the described set are not closed under addition, take e.g.

$$[x^{10}] + [-x^{10} + x] = [x] \notin \mathbb{R}[X]^{10}$$

Thus, $\mathbb{R}[X]^{10}$ *is not a vector space*

Total of marks: 10