Week 2: Functions

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

For which x is $x^2 - 5x + 6 \geq 0$?

(a) $x \leq 2$ and $x \geq 3$ (100%)
(b) $2 \leq x \leq 3$
(c) $x \leq -6$ and $x \geq 1$
(d) $-6 \leq x \leq 1$

Factorizing: $x^2 - 5x + 6$ into $(x-3)(x-2)$ and equating to 0 shows $x = 2$ and $x = 3$ are the roots of the function. By inspection, we find that $x^2 - 5x + 6 \geq 0$ for $x \in (-\infty, 2) \cup (3, \infty)$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $5x^2 = x^4 - 14$.

(a) $x_{1,2} = \pm\sqrt{7}$, $x_{3,4} = \pm i\sqrt{2}$ (100%)
(b) $x_{1,2} = \pm\sqrt{7}$, $x_{3,4} = \pm\sqrt{2}$
(c) $x_{1,2} = \pm\sqrt{2}$, $x_{3,4} = \pm i\sqrt{7}$
(d) $x_{1,2} = \pm\sqrt{2}$, $x_{3,4} = \pm\sqrt{7}$

Rearrange $5x^2 = x^4 - 14$ into $x^4 - 5x^2 - 14 = 0$. Now solve for x^2 using quadratic formula:

$$x^2_{1,2} = \frac{+5 \pm \sqrt{25 + 4 \cdot 14}}{2} = \frac{5 \pm 9}{2} = \{-2, 7\} \Rightarrow$$

$$\xRightarrow{x=\pm\sqrt{x^2}} x_{1,2,3,4} = \{\pm i\sqrt{2}, \pm\sqrt{7}\}$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Solve $x^4 + 4x^2 = 20x + x^3 - 16$ (*Hint:* $x = 2$ satisfies the equation).

(a) $x_{1,2,3,4} = \left\{1, 2, -1 - i\sqrt{7}, -1 + i\sqrt{7}\right\}$ (100%)
(b) $x_{1,2,3,4} = \left\{1, 2, +1 - i\sqrt{7}, +1 + i\sqrt{7}\right\}$
(c) $x_{1,2,3,4} = \left\{1, 1, -1 - i\sqrt{7}, -1 + i\sqrt{7}\right\}$
(d) $x_{1,2,3,4} = \left\{2, 2, -1 - i\sqrt{7}, -1 + i\sqrt{7}\right\}$

Using the hint:

$$
\begin{array}{l}
\big(\;\; x^4 \;\; - x^3 + 4x^2 - 20x + 16\big) \div \big(x-2\big) = x^3 + x^2 + 6x - 8 \\
\underline{-x^4 + 2x^3} \\
\qquad\quad x^3 + 4x^2 \\
\qquad \underline{-x^3 + 2x^2} \\
\qquad\qquad 6x^2 - 20x \\
\qquad\quad \underline{-6x^2 + 12x} \\
\qquad\qquad\qquad -8x + 16 \\
\qquad\qquad\qquad\quad \underline{8x - 16} \\
\qquad\qquad\qquad\qquad\quad 0
\end{array}
$$

Noticing that $x = 1$ is also a solution:

$$
\begin{array}{l}
\big(\;\; x^3 \;\; + x^2 + 6x - 8\big) \div \big(x-1\big) = x^2 + 2x + 8 \\
\underline{-x^3 \;\; + x^2} \\
\qquad\quad 2x^2 + 6x \\
\qquad \underline{-2x^2 + 2x} \\
\qquad\qquad 8x - 8 \\
\qquad\quad \underline{-8x + 8} \\
\qquad\qquad\quad 0
\end{array}
$$

Finally, using quadratic formula: $x = \dfrac{-2 \pm \sqrt{4 - 4 \cdot 8}}{2} = \dfrac{-2 \pm 2\sqrt{-7}}{2} = -1 \pm i\sqrt{7}$

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which function has the roots: $x_{1,2,3,4} = \{-4, 5, 7, 1\}$?

(a) $f(x) = x^4 - 9x^3 - 5x^2 + 153x - 140$ (100%)
(b) $f(x) = x^4 + 5x^3 - 33x^2 - 113x + 140$
(c) $f(x) = x^4 + 7x^3 - 21x^2 - 167x - 140$
(d) $f(x) = x^4 + 17x^3 + 99x^2 + 223x + 140$

Check the values of $f(1), f(-4), f(5), f(7)$ (they all should be 0).

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Given the following function: $f(x) = x^3 + bx^2 + cx - 14$ with $b, c \in \mathbb{R}$ and knowing that $f(1+i) = 0$, solve $f(x) = 0$.

(a) $x_{1,2,3} = \{1+i, 1-i, 7\}$ (100%)
(b) $x_{1,2,3,4} = \{1+i, 1-i, -1-i, -1+i\}$
(c) $x_{1,2,3,4} = \{1+i, 1-i, \pm 7\}$
(d) $x_{1,2,3} = \{1+i, -1+i, -7\}$

$x_1 = 1+i$ is a root of $f(x)$, therefore $x_2 = 1-i$ is as well (f(x) is a polynomial with real coefficients $\Rightarrow$ $f(x) = 0$ implies $f^*(x) = 0^* = f(x^*) = 0$ where * denotes complex conjugation). Note that the product of all roots should be equal to 14 (note that $(x-x_1)(x-x_2)(x-x_3) = x^3 + \alpha x^2 + \beta x - x_1x_2x_3$). $(1-i)(1+i) = 2 \Rightarrow x_3 = 7$

6.

MULTI 1.0 point 0 penalty Single Shuffle

Find all the roots (real or complex) of the polynomial

$$p(x) = 24 - 8\,x - 18\,x^2 + 18\,x^3 - x^4 - 4\,x^5 + x^6\,.$$

Hint: $x = 3$ is a root. Divide out the associated linear factor and continue with more roots that are easy to guess.

(a) $x_{1,2,3,4,5,6} = \{-2, -1, 2, 3, 1+i, 1-i\}$ (100%)
(b) $x_{1,2,3,4,5,6} = \{-3, -1, 2, 3, 1+i, 1-i\}$
(c) $x_{1,2,3,4,5,6} = \{-2, -1, 1, 3, 2+i, 2-i\}$
(d) $x_{1,2,3,4,5,6} = \{-2, -1, 1, 3, 1+i, 1-i\}$

7.

MULTI 1.0 point 0 penalty Single Shuffle

Let $f(x) = e^{-9x+3}$. Determine the Domain and Range of $f(x)$ and its inverse$f^{-1}(x)$.

(a) $Dom(f) = (-\infty, \infty)$, $Ran(f) = (0, \infty)$,
$Dom(f^{-1}) = (0, \infty)$, $Ran(f^{-1}) = (-\infty, \infty)$ (100%)
(b) $Dom(f) = (-\infty, \infty)$, $Ran(f) = [0, \infty)$,
$Dom(f^{-1}) = [0, \infty)$, $Ran(f^{-1}) = (-\infty, \infty)$
(c) $Dom(f) = (0, \infty)$, $Range(f) = (-\infty, \infty)$,
$Dom(f^{-1}) = (-\infty, \infty)$, $Ran(f^{-1}) = (0, \infty)$
(d) $Dom(f) = [0, \infty)$, $Ran(f) = [0, \infty)$,
$Dom(f^{-1}) = [0, \infty)$, $Ran(f^{-1}) = [0, \infty)$

e^x is defined for all real values, and $0 < e^x \leq 1$ for $x \in (-\infty, 0]$, and $1 < e^x < \infty$ for $x \in (0, \infty)$. Since e^x is invertible, then the domain of f^{-1} is the range of f and the range of f^{-1} is the domain of f.

8.

MULTI 1.0 point 0 penalty Single Shuffle

Let $g(x) = e^{-13x^2+7}$. Determine the Domain and Range of $g(x)$.

(a) $Domain(g) = (-\infty, \infty)$, $Range(f) = (0, e^7]$ (100%)
(b) $Domain(g) = [0, \infty)$, $Range(f) = (0, \infty)$
(c) $Domain(g) = (-\infty, \infty)$, $Range(f) = [0, \infty)$
(d) $Domain(g) = [0, \infty)$, $Range(f) = (0, 1]$

e^{-x^2} is well defined for all real values, and it has a maximum at $x = 0$. Thus, its range is $(0, f(0)) = (0, e^7)$.

9.

MULTI 1.0 point 0 penalty Single Shuffle

Consider the two variable equation $(y-2)^2 = 2(x-1)$. Which of the following is true?

(a) y does not describe a function of x and x describes a function of y (100%)
(b) y describes a function of x and x describes a function of y
(c) y describes a function of x and x does not describe a function of y
(d) y does not describe a function of x and x does not describe a function of y

x is simply a quadratic function of y, but $y(x)$ takes two values for every x (namely $\pm\sqrt{2(x-1)}+2$)

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find the sum of the binomial coefficients:

$$\sum_{k=0}^{n} \binom{n}{k}$$

(a) 2^n (100%)
(b) e^n
(c) $(n!)^2$
(d) n^n

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \cdot y^{n-k} \implies (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k \cdot 1^{n-k}$$

$$\implies 2^n = \sum_{k=0}^{n} \binom{n}{k}$$

Total of marks: 10