Week 3: Functions and Limits

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find all horizontal asymptotes of $y(x) = \sin\left(\frac{1}{x}\right) \cdot x^2 + \frac{1}{x-2}$. (Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.)

(a) $\{\}$ (100%)
(b) $\{y = 0\}$
(c) $\{y = 2\}$
(d) $\{y = 0, y = 2\}$

$$\lim_{x\to\pm\infty} y(x) = \pm\infty \Rightarrow \textit{ there are no horizontal asymptotes}$$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is a horizontal asymptote of the function

$$y = \frac{4x}{\log(|x|^7) + 7x}$$

(a) $y = \frac{4}{7}$ (100%)
(b) $y = \frac{4}{7^2}$
(c) $y = 0$
(d) The function has no horizontal asymptote

$$\lim_{x\to\infty} \frac{4x}{\log(|x|^7) + 7x} = \lim_{x\to\infty} \frac{x}{x} \cdot \frac{4}{\frac{\log(|x|^7)}{x} + 7} = \lim_{x\to\infty} \frac{4}{\frac{\log(|x|^7)}{x} + 7} = \frac{4}{7}$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find all vertical asymptotes of $y(x) = \sin\left(\frac{1}{x}\right) \cdot x^2 + \frac{1}{x-2}$.

(a) $\{\}$
(b) $\{x = 0\}$
(c) $\{x = 2\}$ (100%)
(d) $\{x = 0, x = 2\}$

Check potential points of discontinuity - $y(x)$ is continuous on $\mathbb{R}\backslash\{0, 2\}$

$$\lim_{x\nearrow 2} y(x) = -\infty, \lim_{x\searrow 2} y(x) = +\infty, \lim_{x\to 0} y(x) = -\frac{1}{2} \Rightarrow$$

$\Rightarrow$ *there is a horizontal asymptote $x = 2$*

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find an oblique asymptote of $y(x) = \sin\left(\frac{1}{x}\right) \cdot x^2 + \frac{1}{x-2}$. The oblique asymptote $y = ax + b$ can be obtained (if exists) by finding coefficients from the asymptotic behaviour: $a = \lim_{x\to\pm\infty} \frac{y(x)}{x}$, $b = \lim_{x\to\pm\infty} y(x) - ax$. (Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.)

(a) $\{y = 2x + 1/2\}$
(b) $\{y = 2x + 1\}$
(c) $\{y = x\}$ (100%)
(d) $\{y = x - 1/2\}$

Calculate coefficients of the asymptotic line:

$$a = \lim_{x\to-\infty} \frac{y(x)}{x} = \lim_{x\to+\infty} \sin\left(\frac{1}{x}\right) \cdot \frac{1}{1/x} = 1 \Rightarrow$$

$$\Rightarrow b = \lim_{x\to\pm\infty} y(x) - ax = \lim_{x\to\pm\infty} \sin\left(\frac{1}{x}\right) \cdot \frac{1}{1/x} \cdot x + \frac{1}{x-2} - x = 0$$

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Find all horizontal asymptotes of $y(x) = \frac{\left(\ln\left(\frac{1}{x}\right) + \ln(x)\right) \cdot (x^2 + x + 2x) + x + \ln(x)}{x}$

(a) $\{\}$
(b) $\{y = 0\}$
(c) $\{y = 1\}$ (100%)
(d) $\{y = 2\}$

$$\lim_{x\to\pm\infty} \frac{\left(\ln\left(\frac{1}{x}\right) + \ln(x)\right) \cdot (x^2 + x + 2x) + x + \ln(x)}{x} = \lim_{x\to\pm\infty} \frac{x + \ln(x)}{x} = 1$$

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Evaluate the limit:

$$\lim_{v\to 2} \frac{2-v}{\frac{1}{2} - \frac{1}{v}}$$

(a) 4
(b) -4 (100%)
(c) 2
(d) -1

$$\lim_{v\to 2} \frac{2-v}{\frac{1}{2} - \frac{1}{v}} = \lim_{v\to 2} \frac{2-v}{\frac{v-2}{2v}} = \lim_{v\to 2} 2v \cdot \frac{2-v}{-(2-v)} = \lim_{v\to 2} -2v = -4$$

7.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate the limit:

$$\lim_{y\to 0} \frac{\sqrt{2+y}-\sqrt{2-y}}{4y}$$

(a) $\frac{1}{4\sqrt{2}}$ (100%)
(b) $\frac{1}{\sqrt{2}}$
(c) $\sqrt{2}$
(d) 1

$$\begin{aligned}\lim_{y\to 0} \frac{\sqrt{2+y}-\sqrt{2-y}}{4y} &= \lim_{y\to 0} \frac{\sqrt{2+y}-\sqrt{2-y}}{4y} \cdot \frac{\sqrt{2+y}+\sqrt{2-y}}{\sqrt{2+y}+\sqrt{2-y}} = \\ = \lim_{y\to 0} \frac{(2+y)-(2-y)}{4y(\sqrt{2+y}+\sqrt{2-y})} &= \lim_{y\to 0} \frac{(2y)}{4y(\sqrt{2+y}+\sqrt{2-y})} = \\ = \lim_{y\to 0} \frac{1}{4(\sqrt{2+y}+\sqrt{2-y})} &= \frac{2}{4(2\sqrt{2})} = \frac{1}{4\sqrt{2}}\end{aligned}$$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate the limit:

$$\lim_{x\to 0} \left(\sqrt{x}\ln x + e^x x^3\right).$$

(a) 1
(b) 0 (100%)
(c) $+\infty$
(d) $-\infty$

$$\lim_{x\to 0} \left(\sqrt{x}\ln x + e^x x^3\right) = 0 + 1\cdot 0 = 0.$$

First we use the sum rule for limits. Then, $x^\alpha \ln x \to 0$ as $x \to 0$ was shown in class (for any $\alpha > 0$). For the second summand we use the product rule for limits.

9.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate the limit:

$$\lim_{x\to 0} \frac{1-\cos x}{x\tan x}.$$

(Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$.)

(a) 1
(b) 0
(c) 1/2 (100%)
(d) 2

$$\lim_{x\to 0} \frac{1-\cos x}{x\tan x} = \lim_{x\to 0} \frac{2\sin^2(x/2)}{4(x/2)^2} \cdot \frac{x}{\tan x} = \frac{1}{2}$$

(using $\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x}{\tan x} = 1$ *and the product law for limits)*

10.

MULTI 1.0 point 0 penalty Single Shuffle

Let

$$f(x) := \begin{cases} kx + 7 & \text{for } x \geq 2, \\ x^2 + 19 & \text{for } x < 2. \end{cases}$$

For what value of k is $\lim_{x\to 2} f(x)$ defined?

(a) 1
(b) 2
(c) 4
(d) 8 (100%)

The limit from the right is $\lim_{x\to 2^+} f(x) = 2k + 7$*, and the limit from the left is* $\lim_{x\to 2^-} f(x) = 2^2 + 19 = 23$*. The limit exists if left and right limits agree, i.e., if* $2k + 7 = 23$*, meaning* $k = 8$*.*

Total of marks: 10