Week 4: Limits, Continuity, and Start of Derivatives

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following functions does not have a horizontal asymptote

(a) $f(x) = \frac{e^{|x|}}{x^m + x^{m-1} + ... + x + 1}$ for $m \in \mathbb{N}$ (100%)
(b) $f(x) = \frac{\log|x|}{x}$
(c) $f(x) = \frac{\log(|x|^7)}{\log(|x|^3) + x}$
(d) $f(x) = \frac{a_0 + a_1 \cdot x + ... + a_n \cdot x^n}{b_0 + b_1 \cdot x + ... + b_n \cdot x^n}$, where all coefficients are non-zero.

> $\exp(x)$ *grows faster than any polynomial, and thus* $\frac{\exp(x)}{x^m}$ *diverges* $\forall m < \infty$ *as* $x \to \infty$. *Then* $\frac{e^{|x|}}{x^m}$ *diverges as* $x \to \pm\infty$. *The divergence can also be seen by the power series of* e^x*:*
>
> $$\lim_{x\to\infty} \frac{e^{|x|}}{x^m} = \lim_{x\to\infty} \frac{\sum_{k=0}^{\infty} \frac{x^k}{k!}}{x^m} \to \infty$$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Evaluate the limit:

$$\lim_{x\to 0} \frac{12^x - 1}{x}$$

(a) $\ln(12)$ (100%)
(b) $1/\ln(12)$
(c) 12
(d) 0

> $$\lim_{x\to 0} \frac{12^x - 1}{x} = \lim_{y\to 0} \frac{y \cdot \ln 12}{\ln(y+1)} = \text{(use substitution } y = 12^x - 1 \Rightarrow x = \frac{\ln(y+1)}{\ln 12}\text{)}$$
>
> $$= \ln(12) \cdot \lim_{y\to 0} \frac{1}{\ln\left[(1+y)^{1/y}\right]} = \ln 12 \cdot \frac{1}{\ln e} = \ln 12$$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Evaluate the limit:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \frac{i^2}{N^3}$$

(a) 1/3 (100%)

(b) 1/2
(c) 0
(d) 1

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{i^2}{N^3} = \lim_{N\to\infty}\frac{N(N+1)(2N+2)}{6N^3} = \frac{1}{3}$$

(using $\sum_{i=1}^{N} i^2 = \dfrac{N(N+1)(2N+2)}{6}$*, which can be proven by induction)*

4.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate the limit:

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{i^2+i}$$

(a) 1 (100%)
(b) 9/8
(c) 4/3
(d) 2

$$\lim_{N\to\infty}\sum_{i=1}^{N}\frac{1}{i^2+i} = \lim_{N\to\infty}\sum_{i=1}^{N}\left[\frac{1}{i}-\frac{1}{i+1}\right] = \lim_{N\to\infty}\left[\frac{1}{1}-\frac{1}{N+1}\right] = \lim_{N\to\infty}\frac{1}{1+\frac{1}{N}} = 1$$

5.

MULTI 1.0 point 0 penalty Single Shuffle

Check by induction which of the following is true:

(a) $\sum_{k=1}^{n}(2k-1) = n^2$ (100%)
(b) $n! < 2^n$ for $n > 4$
(c) $\sum_{k=1}^{n} 2^{k-1} = 2^n + 1$
(d) $\sum_{k=1}^{n} k^3 = n^2(n+1)^2$

Case n = 1: $2\cdot 1 - 1 = 1^2$
Case $n = k+1$*:*

$$\sum_{k=1}^{n}(2k-1) = (2n-1) + \sum_{k=1}^{n-1}(2k-1) = (2n-1) + (n-1)^2 =$$

$$= 2n - 1 + n^2 - 2n + 1 = n^2$$

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $f(x)$ be a differentiable function. Now consider

$$f_1(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}, \qquad f_2(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

(a) Both f_1 and f_2 define the derivative of f (100%)
(b) f_2 defines a derivative while f_1 does not
(c) f_1 defines a derivative while f_2 does not
(d) Neither f_1 or f_2 define the derivative of f

We can see that the definitions are equivalent, as defining $x_0 = x - h$, we have $x \to x_0$ as $h \to 0$, and

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

which is the definition of the derivative

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Using the limit definition of the derivative and letting $f(x) = x^2 + x$, which of the following is true?

(a) $f'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$ (100%)
(b) $f'(x) = \lim_{h\to 0} \frac{x^2 + h^2 + x + h - x^2 - x}{h}$
(c) $f'(x) = \lim_{h\to 0} \frac{x^2 + x + h - x^2 - x}{h}$
(d) $f'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h + x + h - x^2 - x}{h}$

By definition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^2 + (x+h)] - [x^2 + x]}{h}$$

$$= \lim_{h\to 0} \frac{x^2 - 2xh + h^2 + x + h - x^2 - x}{h}$$

8.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $m \geq 2$. Consider the piecewise function $f(x) = \begin{cases} x^m, & \text{if } x < 0 \\ 0, & \text{if } x \geq 0 \end{cases}$
Evaluate $f'(0)$ using the limit definition of the derivative.

(a) $f'(0) = 0$ (100%)
(b) $f'(0) = 1$

(c) f is not differentiable at $x = 0$
(d) $f'(0) = m - 1$

Since this is a piecewise function we must consider the limits from above and below.

$$\lim_{h \searrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \searrow 0} \frac{h^m - 0}{h} = \lim_{h \searrow 0} h^{m-1} = 0$$

$$\lim_{h \nearrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \nearrow 0} \frac{0 - 0}{h} = \lim_{h \nearrow} 0 = 0$$

Limits agree and thus $f'(0) = 0$

9.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Consider the piecewise function $f(x) = \begin{cases} -x^2, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ \sin x, & \text{if } x > 0 \end{cases}$

Evaluate $f'(0)$ using the limit definition of the derivative.

(a) f is not differentiable at $x = 0$ (100%)
(b) $f'(0) = 1$
(c) $f'(0) = 0$
(d) $f'(0) = \pi$

Since this is a piecewise function we must consider the limits from above and below.

$$\lim_{h \searrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \searrow 0} \frac{\sin(h) - \sin(0)}{h} = \lim_{h \searrow 0} \frac{\sin(h)}{h} = 1$$

$$\lim_{h \nearrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \nearrow 0} \frac{h^2 - 0}{h} = \lim_{h \nearrow 0} h = 0$$

Since derivatives from both sides do not match, the function is not differentiable at $x = 0$.

10.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

The ReLU (Rectified Linear Unit) function is defined as $\text{ReLU}(x) = \max\{0, x\}$ Which of the following is true?

(a) ReLU is differentiable everywhere but $x = 0$, with $\frac{\mathrm{d}}{\mathrm{d}x}\text{ReLU}(x) = \left\{ \begin{array}{ll} 0, & \text{if } x \leq 0 \\ 1, & \text{else} \end{array} \right\}$ (100%)
(b) ReLU is a differentiable function (differentiable everywhere)
(c) ReLU is nowhere differentiable
(d) ReLU is differentiable at finitely many places

The definition of ReLU is analogous to $f(x) = \left\{ \begin{array}{ll} 0, & \textit{if } x \leq 0 \\ x, & \textit{else} \end{array} \right\}$ *The limits from below and above agree everywhere but at* $x = 0$ *for this function.*

Total of marks: 10