Week 6: Optimization

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following functions is guaranteed to have a stationary point ($f'(c) = 0$) by Rolle's Theorem?

(a) $f(t) = 2t^3 - t^2 - t$ (100%)
(b) $f(x) = x^3$
(c) $f(t) = \log(t)$
(d) $f(x) = a^x$

We have that $f(1) = f(0) = 0$. Therefore, Rolle's Theorem states that $\exists c$ such that $f'(c) = 0$.
Note that, even though x^3 does have a stationary point, there are no two distinct points a, b such that $f(a) = f(b)$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Given $f(x) = 2x^3 - 9x^2 - 24x + a$ and knowing that the equation $f(x) = 0$ has 3 distinct real solutions $x_1 < x_2 < x_3 \in \mathbb{R}$ which of the following is always true?

(a) $x_1 \in (-\infty, -1), x_2 \in (-1, 4), x_3 \in (4, \infty)$ (100%)
(b) $x_1 \in (-\infty, -2), x_2 \in (-2, 3), x_3 \in (3, \infty)$
(c) $x_1 \in (-\infty, -1), x_2 \in (-4, 1), x_3 \in (4, \infty)$
(d) $x_1 \in (-\infty, -2), x_2 \in (-3, 2), x_3 \in (3, \infty)$

$f'(x) = 6(x^2 - 3x - 4) = 6(x - 4)(x + 1)$ has the following roots: $\tilde{x}_1 = -1, \tilde{x}_2 = 4$. Roots of $f'(x)$ lie between the roots of $f(x)$ (recall Rolle's theorem), therefore $x_1 < \tilde{x}_1 = -1 < x_2 < \tilde{x}_2 = 4 < x_3$.

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Which of the following is not a requirement for the Mean Value Theorem to hold?

(a) The derivative of f must be continuous on (a, b) (100%)
(b) f must be continuous on $[a, b]$
(c) f must be differentiable on (a, b)
(d) f must be a function from $[a, b]$ to $\mathbb{R}$

The derivative must only be defined on each point $\in (a, b)$, but it does not need to be continuous for MVT to hold.

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $f(x) = \dfrac{x}{1+x}$. What is the value of c over the interval $(0, 3)$ such that the Mean Value Theorem is satisfied?

(a) $c = 1$ (100%)
(b) $c = -3$
(c) $c = \frac{1}{4}$
(d) The MVT does not apply in this case

We have that $f'(x) = \dfrac{(1+x)-x}{(1+x)^2} = \dfrac{1}{(1+x)^2}$.
We want a c *such that* $f'(c) = \dfrac{f(3)-f(0)}{3-0} = \dfrac{1}{3} \cdot \left(\dfrac{3}{4} - 0\right) = \dfrac{1}{4}$
$\implies \dfrac{1}{(1+c)^2} = \dfrac{1}{4} \implies c = -1 \pm \sqrt{4} = -1 \pm 2$, *and only* $1 \in (0,3)$

5.

MULTI 1.0 point 0 penalty Single Shuffle

Find all local minima of $\sin(x)$.

(a) $x = -\frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$ (100%)
(b) $x = -\frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$
(c) $x = \frac{\pi}{2} + 2\pi n$ where $n \in \mathbb{Z}$
(d) $x = \frac{\pi}{2} + \pi n$ where $n \in \mathbb{Z}$

Let $f(x) = \sin(x)$. *Then* $f'(x) = \cos(x) = 0 \Rightarrow x = \frac{\pi}{2} + \pi k$, $k \in \mathbb{Z}$. *To differentiate between minima and maxima:* $f''(x) = -\sin(x)$. $f''(x) < 0$ *for* $x = \frac{\pi}{2} + 2\pi k$, $k \in \mathbb{Z}$ *(corresponds to local maxima),* $f''(x) > 0$ *for* $x = -\frac{\pi}{2} + 2\pi n$, $n \in \mathbb{Z}$ *(corresponds to local minima).*

6.

MULTI 1.0 point 0 penalty Single Shuffle

Close to a straight horizontal river a drone is located at a distance l_1 from it. (If the drone would fly straight to the river, it would meet the river at $x = 0$.) There is also a bonfire at a distance l_2 from the river. The distance between the drone and the bonfire *along the river* is L (i.e., if the bonfire were moved straight to the river, it would meet it at $x = L$). The drone plans to fly along some line to the river (to get to it at a distance x along the river from the initial drone position) to get some water and afterwards move along some line to the bonfire to put it out. To optimize the route, the drone chooses x which minimizes the total travel distance to the bonfire. What does the ratio $\frac{l_1}{x}$ equal to?

(a) $\dfrac{l_1}{x} = \dfrac{l_2}{L-x}$ (100%)
(b) $\dfrac{l_1}{x} = \dfrac{l_2}{x}$
(c) $\dfrac{l_1}{x} = \dfrac{l_2}{\sqrt{L^2 - x^2}}$

(d) $\frac{l_1}{x} = \frac{\sqrt{L^2+x^2}}{l_2}$

The total travel distance is: $d_{\text{total}} = \sqrt{l_1^2+x^2} + \sqrt{l_2^2+(L-x)^2}$.
To minimize it $\frac{\mathrm{d}}{\mathrm{d}x}d_{\text{total}} = \frac{x}{\sqrt{l_1^2+x^2}} - \frac{L-x}{\sqrt{l_2^2+(L-x)^2}} = 0 \Rightarrow$

$$\Rightarrow \frac{x}{\sqrt{l_1^2+x^2}} = \frac{L-x}{\sqrt{l_2^2+(L-x)^2}} \Rightarrow \frac{x}{l_1} = \frac{L-x}{l_2}$$

7.

MULTI 1.0 point 0 penalty Single Shuffle

Which side lengths of a rectangle with a perimeter of $4L$ maximize its area?

(a) (L, L) (100%)
(b) $(\frac{3}{2}L, \frac{1}{2}L)$
(c) $(\frac{1}{4}L, \frac{7}{4}L)$
(d) $(0.01L, 1.99L)$

The area of a general rectangle with side lengths $(x, 2L-x)$ *(this satisfies the perimeter requirement:* $x + x + 2L - x + 2L - x = 4L$*) is* $A = 2Lx - x^2$. *Find its maximum:* $\frac{\mathrm{d}A}{\mathrm{d}x} = 2L - 2x = 0 \Rightarrow x = L$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Find the radius of the circle which minimizes *the sum of the squares of the distances* of the circle centered at the origin to the points $(1, 0)$ and $(-1, 1)$.

(a) $r = \frac{1+\sqrt{2}}{2}$ (100%)
(b) $r = 1$
(c) $r = \sqrt{2}$
(d) $r = \frac{\sqrt{3}-1}{2}$

Let D *be the sum of the square of the distances between each point and the circumference.*

$$d_1^2 = (1-r)^2, d_2^2 = (\sqrt{2}-r)^2$$

$$D = d_1^2 + d_2^2 \implies \frac{\mathrm{d}D(r)}{\mathrm{d}r} = 2(1-r)\cdot(-1) + 2(\sqrt{2}-r)\cdot(-1)$$
$$= -2(1-r) - 2(\sqrt{2}-r)$$

Since we want to minimize $D(r)$*, we have*

$$D'(r) = 0 \implies -2(1-r) - 2(\sqrt{2}-r) = 0 \implies 1 + \sqrt{2} - 2r = 0 \implies r = \frac{1+\sqrt{2}}{2}$$

9.

MULTI 1.0 point 0 penalty Single Shuffle

An algorithm can calculate the universe partition function (a function one can give to a statistical physicist to find out the secrets of our universe) with a precision level N in time $T(N) = a^2/2 - (e^N a \sin(2N))/(\cos(2N) + 1)$. The parameter a corresponds to a specific choice of hyperparameters in the algorithm. For a given $N \in Dom(T)$, which a does one need to choose to use the fastest version of the algorithm?

(a) $a = e^N \tan(N)$ (100%)
(b) $a = \sin(N)\cos(N)e^N$
(c) $a = \dfrac{e^N \ln(N)}{2}$
(d) $a = \dfrac{e^N \sin(x)}{2}$

Find the minimizer:

$$\frac{\mathrm{d}T}{\mathrm{d}a} = a - e^N \frac{\sin(2N)}{\cos(2N)+1} = 0 \Rightarrow a = e^N \frac{2\sin(N)\cos(N)}{2\cos(N)^2} = e^N \tan(N)$$

10.

MULTI 1.0 point 0 penalty Single Shuffle

The potential energy of a thin wooden rod hanging partially in water can be modelled as follows:

$$E_{\text{pot}} = -\frac{\rho_w g V L \cos(\varphi)}{2} + \rho_0 g \left(1 - \frac{h}{L\cos(\varphi)}\right)^2 \frac{V L \cos(\varphi)}{2}$$

where ρ_w, ρ_0 are wood and water densities, V, L are the volume and length of the rod, g is a gravitational acceleration constant, h is a constant which fixes one of the rod ends, φ can be varied to change the rod's position. What is the wood density in the equilibrium position (assume water density to be known)? Note that in the stable equilibrium position the following equality holds: $h = \frac{3}{4} L\cos(\varphi)$ (a quarter of the rod is underwater). The equilibrium position can be found by minimizing the potential energy E_{pot} (by finding φ which minimizes E_{pot}).

(a) $\rho_w = \frac{7}{16}\rho_0$ (100%)
(b) $\rho_w = \frac{3}{4}\rho_0$
(c) $\rho_w = \frac{3}{8}\rho_0$
(d) $\rho_w = \frac{5}{28}\rho_0$

Minimize E_{pot} to find the stable equilibrium position:

$$\frac{\mathrm{d}E_{\text{pot}}}{\mathrm{d}\varphi} = \frac{\rho_w g V L \sin(\varphi)}{2} +$$

$$+\frac{\rho_0}{2} g V L \left(2\left(1 - \frac{h}{L\cos(\varphi)}\right)\cos(\varphi)\left(\frac{h}{L}\ \frac{-\sin(\varphi)}{\cos^2(\varphi)}\right) - \left(1 - \frac{h}{L\cos(\varphi)}\right)^2 \sin(\varphi)\right) = 0$$

$$\Rightarrow \rho_w = \rho_0 \left(2\left(1 - \frac{h}{L\cos(\varphi)}\right)\frac{h}{L\cos(\varphi)} + \left(1 - \frac{h}{L\cos(\varphi)}\right)^2\right) = \frac{7}{16}\rho_0$$

Total of marks: 10