Week 7: Extreme Values, Integration

1.

MULTI 1.0 point 0 penalty Single Shuffle

What are the maxima and minima of the function $f(x) = \frac{1}{3}x^3 - \frac{7}{2}x^2 + 10x + 3$?

(a) At $x = 2$ there is a maximum, at $x = 5$ a minimum. (100%)
(b) At $x = 2$ there is a minimum, at $x = 5$ a maximum.
(c) There are no maxima or minima.
(d) At $x = 3$ there is a minimum, at $x = 1$ a maximum.

The derivative is $f'(x) = x^2 - 7x + 10 = (x-2)(x-5)$, i.e., its zeros are at $x = 2$ and $x = 5$. The second derivative is $f''(x) = 2x - 7$. Thus, $f''(2) = -3 < 0$, i.e., at $x = 2$ there is a maximum, and $f''(5) = 3 > 0$, i.e., at $x = 5$ there is a minimum.

2.

MULTI 1.0 point 0 penalty Single Shuffle

For which interval is $f(x) = \frac{x^2}{\pi^2 - x^2}$ positive?

(a) $x \in (-\pi, \pi)$ (100%)
(b) $x \in (-\infty, 0)$
(c) $x \in (-\pi, 0) \cup (\pi, \infty)$
(d) $x \in (-\infty, -\pi) \cup (\pi, \infty)$

f has a root at $x = 0$, and asymptotes at $x = \pm\pi$. Since f is continuous everywhere but at the asymptotes, we only need to examine the regions separated by the asymptotes and the root. By inspection, we can see $f > 0$ for $x \in (-\pi, \pi)$.

3.

MULTI 1.0 point 0 penalty Single Shuffle

For which values of x does $f(x) = -\ln(x) + \sqrt{x}$ have maxima or minima?

(a) $x = 4$ is a minimum (100%)
(b) $x = 2$ is a maximum
(c) $x = -2$ is a maximum and $x = 2$ is a minimum
(d) No maxima or minima

We have:

$$f'(x) = -\frac{1}{x} + \frac{1}{2}x^{-\frac{1}{2}}$$

Assuming $x \neq 0$ *for* $\frac{1}{x}$ *to be defined and* $x > 0$ *for* $\sqrt{x}$*:*

$$f'(x) = 0 \Rightarrow \frac{1}{x} = \frac{1}{2}x^{-\frac{1}{2}} \Rightarrow \sqrt{x} = 2 \Rightarrow x = 4$$

The second derivative is

$$f''(x) = \frac{1}{x^2} - \frac{1}{4}x^{-\frac{3}{2}} \Rightarrow f''(4) = \frac{1}{16} - \frac{1}{32} > 0 \Rightarrow \text{ it is a minima}$$

4.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

For which value of x does $f(x) = 2\mathrm{e}^{-4/x}$ have a point of inflection?

(a) $x = 2$ (100%)
(b) $x = 32$
(c) $x = -4$
(d) There is no point of inflection

$$f'(x) = 2\mathrm{e}^{-4/x} \cdot \frac{4}{x^2} = 8\frac{\mathrm{e}^{-4/x}}{x^2}$$

$$f''(x) = 32\frac{\mathrm{e}^{-4x}}{x^4} + 8\mathrm{e}^{-4/x} \cdot (-2x^3) = \frac{32\mathrm{e}^{-\frac{4}{x}}}{x^4} - \frac{16\mathrm{e}^{-\frac{4}{x}}}{x^3}$$

Assuming $x \neq 0$*:*

$$f''(x) = 0 \Rightarrow \frac{32\mathrm{e}^{-\frac{4}{x}}}{x^4} - \frac{16\mathrm{e}^{-\frac{4}{x}}}{x^3} = 0 \Rightarrow \frac{32}{x} = 16 \Rightarrow x = 2$$

5.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Evaluate $\int \frac{\cos(\pi/x)}{x^2}\,\mathrm{d}x$. (*Hint*: substitute $\frac{\pi}{x}$.)

(a) $-\frac{1}{\pi}\sin\frac{\pi}{x} + C$ (100%)
(b) $-\frac{1}{\pi}\sin \pi x + C$
(c) $\frac{1}{\pi}\sin\frac{1}{x} + C$
(d) $\sin\frac{\pi}{x} + C$

$$\int \frac{\cos(\pi/x)}{x^2}\,\mathrm{d}x =$$

Using substitution: $u := \frac{\pi}{x} \Rightarrow \mathrm{d}u = -\frac{\pi}{x^2}\mathrm{d}x \Rightarrow \frac{\mathrm{d}x}{x^2} = -\frac{\mathrm{d}u}{\pi}$

$$= -\frac{1}{\pi}\int \cos u\,\mathrm{d}u = -\frac{1}{\pi}\sin u + C = -\frac{1}{\pi}\sin\frac{\pi}{x} + C$$

6.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Compute $\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x$. *Hint*: How about a substitution involving the sine?

(a) $\sin^{-1}\left(\frac{x}{3}\right) + C$ (100%)
(b) $\cos^{-1}\left(\frac{x}{3}\right) + C$
(c) $\sin\left(\frac{x}{3}\right) + C$
(d) $2\sqrt{9-x^2} + C$

A good substitution is $x = 3\sin u$. *Then* $\mathrm{d}x = 3\cos u\,\mathrm{d}u$. *Then*

$$\int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x = \int \frac{1}{\sqrt{9-(3\sin u)^2}}3\cos(u)\,\mathrm{d}u = \int \frac{1}{3\cos u}3\cos(u)\,\mathrm{d}u$$

$$= \int 1\,\mathrm{d}u = u + C.$$

Substituting back yields the result.

7.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Evaluate $\int \sin(x)\ln(\cos x)\,\mathrm{d}x$ (*Hint:* use integration by parts)

(a) $\cos x(1-\ln\cos x) + C$ (100%)
(b) $\cos x(1-\ln\cos x)$
(c) $\cos x(1+\ln\cos x)$
(d) $\cos x(1+\ln\cos x) + C$

$$\int \sin(x)\ln(\cos x)\,\mathrm{d}x = -\cos x\ln(\cos x) - \int(-\cos x)\frac{-\sin x}{\cos x}\,\mathrm{d}x =$$

$$= -\cos x\ln(\cos x) - \int \sin x\,\mathrm{d}x = \cos x(1-\ln\cos x) + C$$

8.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Compute $\int x^n e^x \mathrm{d}x$ for $n \in \mathbb{N}$.

(a) $\left(\sum_{k=0}^{n}(-1)^k \frac{n!\,x^{n-k}}{(n-k)!}\right) \mathrm{e}^x$ (100%)

(b) $\left(\sum_{k=0}^{n} \frac{n!\,x^{n-k}}{(n-k)!}\right) \mathrm{e}^x$

(c) $\left(\sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!}\right) \mathrm{e}^x$

(d) $\left(\sum_{k=0}^{n} k!\,x^{n-k}\right) \mathrm{e}^x$

Using integration by parts:
$\int x^n e^x = x^n e^x - \int n x^{n-1} e^x = x^n e^x - n x^{n-1} e^x + \int n(n-1)x^{n-2} e^x = \cdots = \left(\sum_{k=0}^{n}(-1)^k \frac{n!\,x^{n-k}}{(n-k)!}\right) \mathrm{e}^x$

9.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate $\int \sec^2(x)\tan(x)\mathrm{d}x$.

Hint: Try first computing $\frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x$; $\left(\sec(x) := \frac{1}{\cos(x)}\right)$

(a) $\frac{\sec^2(x)}{2} + C$ (100%)

(b) $\frac{\sin^2(x)}{2} + C$

(c) $\frac{1-\tan(x)}{2} + C$

(d) $\frac{\cos(\sin(x))}{2} + C$

First: $\frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\cos^2(x)} = -\frac{1}{\cos^4(x)} \cdot (-2)\sin(x)\cos(x) = 2\frac{\sin(x)}{\cos(x)}\frac{1}{\cos^2(x)} = 2\tan(x)\sec^2(x)$.
Thus, we find $\int \sec^2(x)\tan(x)\mathrm{d}x = \frac{1}{2}\int\left(\frac{\mathrm{d}}{\mathrm{d}x}\sec^2 x\right)\mathrm{d}x = \frac{1}{2}\sec^2 x + C$.

10.

MULTI 1.0 point 0 penalty Single Shuffle

Given $I_n = \int_0^1 (a - bx^3)^n$, find a relationship between I_n and I_{n-1}.

(a) $I_n = \frac{3n}{3n+1} I_{n-1}$ (100%)

(b) $I_n = \frac{3n}{3n-1} I_{n-1}$

(c) $I_n = \left(\dfrac{n}{n+1}\right)^3 I_{n-1}$

(d) $I_n = \dfrac{n}{n+1} I_{n-1}^3$

We can separate the integral as:

$$\int_0^1 (1-x^3)^n \mathrm{d}x = \int_0^1 (1-x^3)(1-x^3)^{n-1} \mathrm{d}x = \int_0^1 (1-x^3)^{n-1} \mathrm{d}x - \int_0^1 x \cdot x^2 (1-x^3)^{n-1} \mathrm{d}x$$

Using integration by parts:

$$= I_{n-1} + \left[\frac{x}{3n}(1-x^3)^n\right]_0^1 - \int_0^1 \frac{1}{3n}(1-x^3)^n \mathrm{d}x = I_{n-1} - \frac{1}{3n} I_n$$

$$\Rightarrow I_n = \frac{3n}{3n+1} I_{n-1}$$

Total of marks: 10