Week 8: Integrating Rational Functions, and Definite integrals

1.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Calculate $\int_0^1 \frac{1}{x^2+14x+98}\,\mathrm{d}x$

(a) $\frac{1}{7}\left[\arctan\left(\frac{8}{7}\right) - \frac{\pi}{4}\right]$ (100%)
(b) $\frac{1}{28}\left[\pi - \arcsin\left(\frac{5}{4}\right)\right]$
(c) 1
(d) 3π

$$\int_0^1 \frac{1}{x^2+14x+98}\,\mathrm{d}x = \int_0^1 \frac{1}{(x+7)^2+49}\,\mathrm{d}x =$$
$$= \frac{1}{7}\arctan\left(\frac{x+7}{7}\right)\Big|_0^1 = \frac{1}{7}\left[\arctan\left(\frac{8}{7}\right) - \pi\right]$$

2.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Let $P(t)$ denote the number of bacteria in a sample at time t. Initially, $P(0) = 100$ and it increases at a rate $\frac{\mathrm{d}P}{\mathrm{d}t} = 20e^{3t}$. What is the population at $t = 50$?

(a) $P(50) \approx 9.3 \times 10^{65}$ (100%)
(b) $P(50) \approx 2.8 \times 10^{66}$
(c) $P(50) \approx 8.4 \times 10^{66}$
(d) $P(50) \approx 3.5 \times 10^{22}$

Integrating the rate of change, we obtain $P(t) = \frac{20}{3}e^{3t} + C$. *Given* $P(0) = 100$, *then* $P(t) = 100 + \frac{20}{3}(e^{3t} - 1)$. *Then* $P(50) \approx 9.3 \times 10^{65}$

3.

MULTI | 1.0 point | 0 penalty | Single | Shuffle

Calculate $\int_2^1 \frac{2y^3 - 6y^2}{y^2}\,\mathrm{d}t$

(a) 3 (100%)
(b) -3
(c) 9
(d) -9

$$\int_2^1 \frac{2y^3 - 6y^2}{y^2}\,\mathrm{d}t = \int_2^1 (2y-6)\,\mathrm{d}t = \left[\frac{1}{2}2y^2 - 6y\right]_2^1 = 1^2 - 6(1) - 2^2 + 6(2) = 3$$

4.

MULTI 1.0 point 0 penalty Single Shuffle

Calculate $\displaystyle\int_0^{1/2} \frac{2x^2+2}{x^2-1}\,\mathrm{d}x$

(a) $1-2\ln(3)$ (100%)
(b) $2\ln(3)$
(c) -1
(d) $2\ln(2)-1$

First, polynomial long division yields $\dfrac{2x^2+2}{x^2-1} = 2 + \dfrac{4}{x^2-1}$. *The decomposition into partial fractions yields* $\dfrac{4}{x^2-1} = \dfrac{2}{x-1} - \dfrac{2}{x+1}$. *Thus,*

$$\int_0^{1/2} \frac{2x^2+2}{x^2-1}\,\mathrm{d}x = \int_0^{1/2} 2\,\mathrm{d}x + \int_0^{1/2} \frac{2}{x-1}\,\mathrm{d}x - \int_0^{1/2} \frac{2}{x+1}\,\mathrm{d}x =$$

$$= \Big[2x + 2\ln|x-1| - 2\ln|x+1|\Big]_0^{1/2} = 1 + 2\left(\ln(1/2) - \ln(3/2)\right) =$$

$$= 1 - 2\ln(3)$$

5.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate $\displaystyle\int_0^{1} \frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}\,\mathrm{d}x$

(a) $2\ln(2)$ (100%)
(b) $\ln(3)$
(c) $-\ln(5)$
(d) $4\ln(2)-2$

$$\int_0^1 \frac{3x^2+12x+11}{(x+1)(x+2)(x+3)}\,\mathrm{d}x = \int_0^1 \frac{1}{x+1}\,\mathrm{d}x + \int_0^1 \frac{1}{x+2}\,\mathrm{d}x + \int_0^1 \frac{1}{x+3}\,\mathrm{d}x =$$

$$= \ln(x+1) + \ln(x+2) + \ln(x+3)\Big|_0^1 = \ln(4) - \ln(1) = 2\ln(2)$$

6.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate $\displaystyle\int \frac{e^x}{e^{2x}-e^x}\,\mathrm{d}x$

(a) $\ln\left|\dfrac{e^x-1}{e^x}\right| + C$ (100%)
(b) $\ln\left|\dfrac{e^x-1}{e^x}\right|$

(c) $-\ln(e^{2x}-e^{x})$
(d) $4-\ln(e^{2x}-e^{x})+C$

Letting $u=e^x$ we have

$$I=\int \frac{e^x}{e^{2x}-e^x}\,\mathrm{d}x=\int \frac{\mathrm{d}u}{u^2-u}=\int \frac{\mathrm{d}u}{u(u-1)}=\int\left(\frac{A}{u}+\frac{B}{u-1}\right)\mathrm{d}u$$

$$\Rightarrow A(u-1)+Bu=1\Rightarrow A=-B, A=-1$$

$$\Rightarrow I=\int\left(\frac{-1}{u}+\frac{1}{u-1}\right)\mathrm{d}u=\ln\left|\frac{u-1}{u}\right|+C=\ln\left|\frac{e^x-1}{e^x}\right|+C$$

7.

MULTI 1.0 point 0 penalty Single Shuffle

Evaluate the length of the line $y(x)$ knowing that $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{(1-\cos(x)\sin(x))^2}{\cos^4(x)}e^{-2x}-1}$ on the interval $x\in[0,1]$ using the following integral: $L=\int_{p_0}^{p_1}\mathrm{d}s=\int_0^1\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$.
Hint: To compute the resulting integral, look at the integrand carefully and recall the derivative of $\tan x$.

(a) $\tan(1)/e$ (100%)
(b) $\pi/\tan(e)$
(c) $\pi-\tan(0.5)$
(d) e^{π}

$$L=\int_0^1\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x=\int_0^1\frac{(1-\cos(x)\sin(x))}{\cos^2(x)}e^{-x}\mathrm{d}x=$$

$$=\int_0^1\frac{e^x\cos^{-2}(x)-\tan(x)e^x}{e^{2x}}=\left(\frac{\tan(x)}{e^x}\right)\Bigg|_0^1=\frac{\tan(1)}{e}$$

8.

MULTI 1.0 point 0 penalty Single Shuffle

Calculate $\int_{-1}^{1}f(x)\,\mathrm{d}x$ where

$$f(x)=x\left(\frac{e^x-e^{-x}}{2}\right)\tan(x)$$

(a) $-\pi$
(b) 0 (100%)
(c) e
(d) 2

First note that $f(x)$ is well-defined and continuous on $[-1, 1]$. Now observe that $f(x)$ is an odd function, i.e., $f(x) = f(-x)$. Thus, the integral of $f(x)$ from -1 to 1 is zero.

9.

MULTI 1.0 point 0 penalty Single Shuffle

Find $\int_1^{e} \frac{\ln x}{x}\,dx$ (*Hint:* use a substitution)

(a) 0.5 (100%)
(b) 1.5
(c) 1
(d) 0.75

$$\int_1^{e} \frac{\ln x}{x}\,dx =$$

$$\textit{Using substitution: } u := \ln x \Rightarrow du = \frac{dx}{x}$$

$$= \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}$$

10.

MULTI 1.0 point 0 penalty Single Shuffle

Calculate $\int_0^{\pi/2} x\sin(x)\cos(x)\,dx$ (*Hint:* simplify $\sin(x)\cos(x)$ using a trigonometric identity, and then use integration by parts.)

(a) $\pi/8$ (100%)
(b) $\pi/4$
(c) 0
(d) $3\pi/8$

$$\int_0^{\pi/2} x\sin(x)\cos(x)\,dx = \int_0^{\pi/2} \frac{1}{2}x\sin(2x)\,dx = -\frac{1}{4}x\cos(2x)\Big|_0^{\pi/2} + \frac{1}{4}\int_0^{\frac{\pi}{2}} \cos(2x)\,dx$$

$$= \frac{\pi}{8} + \frac{1}{8}\sin(2x)\Big|_0^{\pi/2} = \frac{\pi}{8}$$

Total of marks: 10