Jacobs University
Fall 2022

January 30, 2023

Operations Research

Final Exam (Make-up)

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

Name: Solutions

Matric. No.: ______________________

Problem 1: Graphical Method [25 points]

Consider the following Linear Programming problem: Maximize

$$Z = 3x_1 + 5x_2$$

subject to

$$\begin{aligned} x_1 &\le 8\,, \\ -2\,x_1 + 5\,x_2 &\le 10\,, \\ 2\,x_1 + 3\,x_2 &\le 22\,, \\ x_1, x_2 &\ge 0\,. \end{aligned}$$

(8) (a) Solve the problem with the graphical method, i.e., draw the feasible region, and find the optimal solution and the corresponding value of Z graphically.

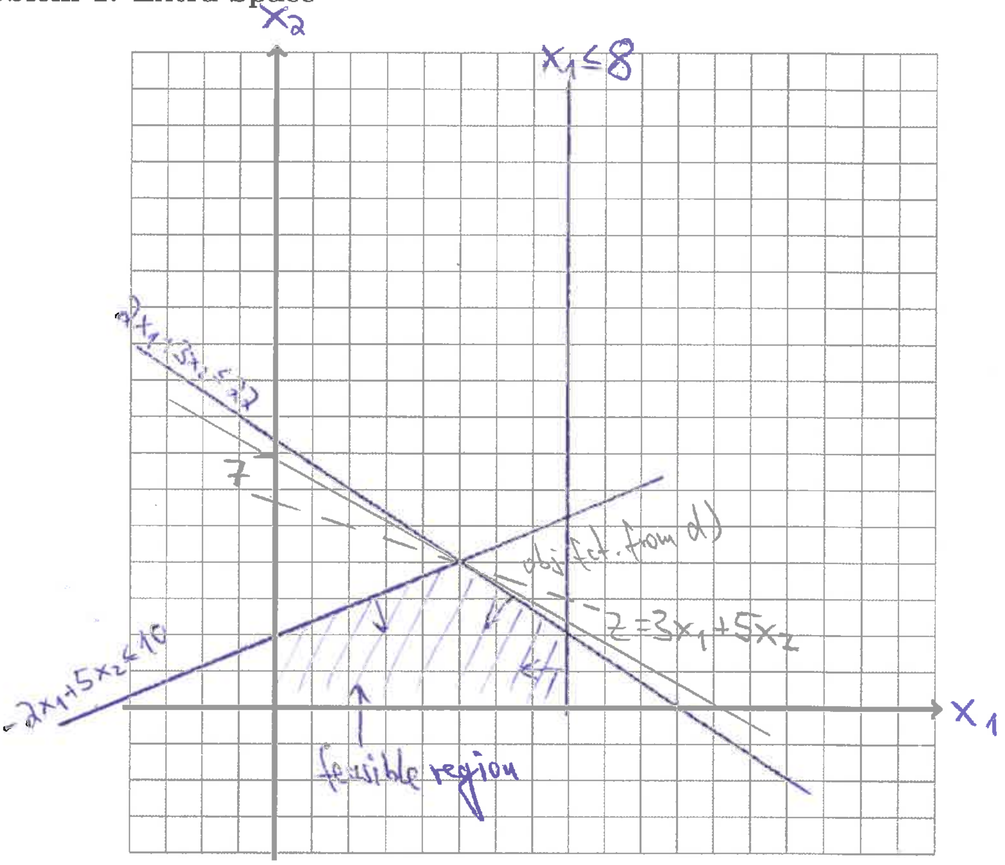
(6) (b) From part (a), identify the two binding constraints. Using this information, compute the optimal solution and the corresponding value of Z.

(3) (c) Now suppose the constraint $x_1 \le 8$ is removed. How does the optimal solution change? Explicitly state the new optimal x_1, x_2 and the corresponding optimal value of Z.

(3) (d) Now suppose the objective function is changed to $Z = x_1 + 3x_2$. (All constraints above remain.) How does the optimal solution change? (Explicitly state the new optimal x_1, x_2 and the corresponding optimal value of Z.)

(5) (e) Now write down the dual LP problem. Without solving the dual problem explicitly, what is the value of $b^{\mathrm{T}}y$, where $b^{\mathrm{T}} = (8, 10, 22)$, and y is the optimal solution of the dual problem?

Problem 1: Extra Space



a) From the picture, we see that the optimal solution is $\approx (x_1, x_2) = (5,4)$, with $z \approx 7 \cdot 5 = 35$.

b) The second and third constraints are binding. The optimal solution follows from solving

$$\begin{aligned} -2x_1 + 5x_2 &= 10 \\ 2x_1 + 3x_2 &= 22 \end{aligned} \Rightarrow 8x_2 = 32 \Rightarrow x_2 = 4 \Rightarrow x_1 = 5.$$

$\Rightarrow z = 3 \cdot 5 + 5 \cdot 4 = 35.$

c) According to the picture the optimal solution does not change. It remains at $(5,4)$, with $z = 35$.

d) The optimal solution is still $(5,4)$, but now with $z = 5 + 3 \cdot 4 = 17$.

e) Dual problem: Minimize $8y_1 + 10y_2 + 22y_3$, subject to $y_1 - 2y_2 + 2y_3 \ge 3$

$5y_2 + 3y_3 \ge 5$

$y_1, y_2, y_3 \ge 0.$

By strong duality, $b^T y = c^T x = 35$ here.

Problem 2: Standard Form and Simplex Method [25 points]

(9) (a) Consider the following Linear Programming problem: Maximize

$$Z = 2x_1 + x_2 - 3x_3$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 40, \\ 4x_1 + x_2 &\leq 100, \\ x_1 + x_2 + x_3 &\geq 20, \\ x_1, x_2 &\geq 0, \end{aligned}$$

and no further restriction on the sign of x_3. Write this problem in standard form, i.e., as the problem to minimize

$$\tilde{Z} = c^{\mathrm{T}} \tilde{x}$$

subject to

$$A\tilde{x} = b, \quad \tilde{x} \geq 0.$$

Specify exactly the vectors b, c and the matrix A.

(16) (b) Consider now the following simpler Linear Programming problem: Maximize

$$Z = 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 &\leq 40, \\ 4x_1 + x_2 &\leq 100, \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solve the problem using the simplex method. In the end, specify the optimal solution and the value of the objective function at the optimal solution.

Problem 2: Extra Space

a) We replace $x_3 = u - v$ with $u, v \geq 0$.

$\Rightarrow$ Minimize $\hat{z} = -z = -2x_1 - x_2 + 3u - 3v$.

The constraints with slack variables become:

$$
\begin{array}{llll}
x_1 + x_2 & + s_1 & & = 40 \\
4x_1 + x_2 & & + s_2 & = 100 \\
-x_1 - x_2 - u + v & & + s_3 & = -20
\end{array}
$$

$$x_1, x_2, u, v, s_1, s_2, s_3 \geq 0.$$

With $\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ u \\ v \\ s_1 \\ s_2 \\ s_3 \end{pmatrix}$, we thus have $c = \begin{pmatrix} -2 \\ -1 \\ 3 \\ -3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 1 \end{pmatrix}$,

$b = \begin{pmatrix} 40 \\ 100 \\ -20 \end{pmatrix}$.

b) The corresponding simplex tableau is:

$$
\left[\begin{array}{cccc|c}
1 & 1 & 1 & 0 & 40 \\
4 & 1 & 0 & 1 & 100 \\
\hline
-2 & -1 & 0 & 0 &
\end{array}\right]
\Rightarrow
\begin{array}{l}
R_1 - \frac{1}{4}R_2 \to R_1 \\
\frac{1}{4}R_2 \to R_2 \\
R_3 + \frac{1}{2}R_2 \to R_3
\end{array}
\left[\begin{array}{cccc|c}
0 & \frac{3}{4} & 1 & -\frac{1}{4} & 15 \\
1 & \frac{1}{4} & 0 & \frac{1}{4} & 25 \\
\hline
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 50
\end{array}\right]
$$

$$
\Rightarrow
\begin{array}{l}
\frac{4}{3}R_1 \to R_1 \\
R_2 - \frac{1}{3}R_1 \to R_2 \\
R_3 + \frac{2}{3}R_1 \to R_3
\end{array}
\left[\begin{array}{cccc|c}
0 & 1 & \frac{4}{3} & -\frac{1}{3} & 20 \\
1 & 0 & -\frac{1}{3} & \frac{1}{3} & 20 \\
\hline
0 & 0 & \frac{2}{3} & \frac{1}{3} & 60
\end{array}\right]
$$

$\Rightarrow$ opt. sol. is $x_1 = 20$, $x_2 = 20$, with $z = 60$.

Problem 3: Dynamic Programming [25 points]

You are traveling home for the holidays. You have bought a number of presents for family and friends, listed in the table below. In your baggage, you have an allowance of 5 kg left. Which presents should you take along and which should you leave for another time if you intend to maximize the total value taken home?

Item	1	2	3	4	5
Value [EUR]	20	5	25	10	20
Weight [kg]	2	1	4	1	3

Solve this problem by dynamic programming. *(Note: You will not get points for simply stating the correct solution; the exercise here is to solve the problem with the dynamic programming method.)*

Stages $i = 1, \dots, 5$ correspond to decision to pack items

We def: • s = remaining weight

• $f_i(s, x_i)$ = value at stage $\geq i$ after decision $x_i = \begin{cases} 0, & \text{leave} \\ 1, & \text{take} \end{cases}$

$i=5$:

s	f_5^*	x_5^*
0,1,2	0	0
3,4,5	20	1

$i=4$:

s	$f_4(s,0)$	$f_4(s,1)$	f_4^*	x_4^*
0	0+0	/	0	0
1	0+0	10+0	10	1
2	0+0	10+0	10	1
3	0+20	10+0	20	0
4	0+20	10+20	30	1
5	0+20	10+20	30	1

$i=3$:

s	$f_3(s,0)$	$f_3(s,1)$	f_3^*	x_3^*
0	0+0	/	0	0
1	0+10	/	10	0
2	0+10	/	10	0
3	0+20	/	20	0
4	0+30	25+0	30	0
5	0+30	25+10	35	1

$i=2$:

s	$f_2(s,0)$	$f_2(s,1)$	f_2^*	x_2^*
0	0+0	/	0	0
1	0+10	5+0	10	0
2	0+10	5+10	15	1
3	0+20	5+10	20	0
4	0+30	5+20	30	0
5	0+35	5+30	35	0 or 1

$i=1$:

s	$f_1(s,0)$	$f_1(s,1)$	f_1^*	x_1^*
5	0+35	20+20	40	1

$\Rightarrow$ Take items 1 and 5, for a total value of 40 € (and total weight of 5 kg.)

Problem 3: Extra Space

Problem 4: Advanced Topics [25 points]

(8) (a) Consider the decision analysis problem with the following payoff table:

Alternative	State of Nature		
	S_1	S_2	S_3
A_1	−80	10	50
A_2	−10	20	40
A_3	10	10	60
Prior Probability	0.2	0.3	0.5

Which alternative should be chosen? What is the resulting expected payoff?

(9) (b) Let us consider a basic inventory management problem with the following assumptions:

- The setup cost per order is called K; The cost per unit is called c;
- The holding (storage) cost is h per unit per time in inventory;
- There is a constant withdrawal rate of m units per time;
- We do not allow for shortages;
- The inventory level is continuously checked (continuous review).

Find the optimal order quantity Q^* that minimizes the cost per time.

(Hint: This exercise is to derive the EOQ formula. Proceed as we did in class, i.e.: Determine the cost per cycle (a picture might be helpful), then the total cost per time, and then find the minimum.)

(8) (c) We consider the following nonlinear optimization problem: Maximize

$$Z = x_1 + x_2$$

subject to

$$\begin{aligned} -x_1^2 + x_2 &\leq 0\,, \\ x_1 &\leq 3\,, \\ x_1, x_2 &\geq 0\,. \end{aligned}$$

(i) Use the graphical method to solve this problem.

(ii) Is this a convex optimization problem? (Briefly justify your answer.) What would convexity or non-convexity imply for solvers such as ipopt (that we discussed in class)?

Problem 4: Extra Space

a) $\mathbb{E}(A_1) = 0.2\cdot(-80) + 0.3\cdot 10 + 0.5\cdot 50 = -16+3+25 = 12$

$\mathbb{E}(A_2) = 0.2(-10) + 0.3\cdot 20 + 0.5\cdot 40 = -2+6+20 = 24$

$\mathbb{E}(A_3) = 0.2\cdot 10 + 0.3\cdot 10 + 0.5\cdot 60 = 2+3+30 = 35$

$\Rightarrow$ Alternative A_3 should be chosen, with largest expected payoff of 35.

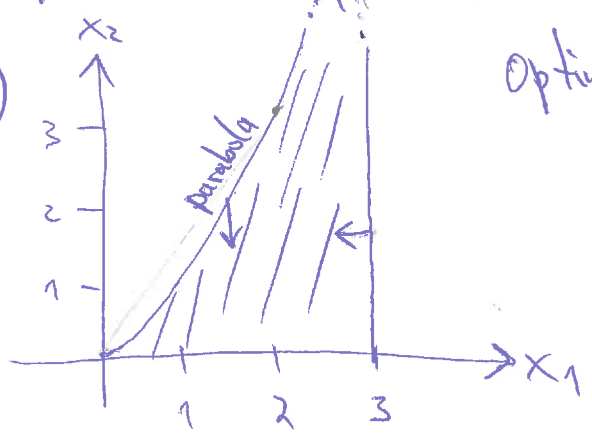
b) Cost per cycle is $C_{cycle} = K + cQ + h\frac{Q}{2}T$, with $T = \frac{Q}{m}$, where Q = quantity ordered.

$\Rightarrow$ Cost per time $C = \frac{C_{cycle}}{T} = \frac{K + cQ + h\frac{Q}{2}T}{T} = \frac{Km}{Q} + cm + \frac{1}{2}hQ$

Minimum: $0 = \frac{dC}{dQ} = \frac{-Km}{Q^2} + \frac{1}{2}h \Rightarrow \boxed{Q^* = \sqrt{\frac{2Km}{h}}}$ is the optimal order quantity.

c)

(i)



Optimal sol. at $x_1 = 3$, $x_2 = 9$, with $Z = 12$

(ii) Here, the feasible region is not convex, i.e., this is not a convex optimization problem. This implies that solvers such as ipopt might get stuck in a local max., but not find the global max.

Problem 5: Pyomo [25 points]

The pyomo program on the last exam page shows an implementation of a minimum cost flow problem (which we discussed in class).

(5) (a) Draw the corresponding network representation of the problem. (Make sure to include the given values of the parameters from the pyomo program.)

(5) (b) Write down the Linear Programming problem that is solved in mathematical notation (i.e., write down the objective function and the constraints).

(5) (c) Briefly explain the meaning of the parameters b, C, U in the context of minimum cost flow problems. In particular, explain what positive, negative, or zero values for entries in b mean.

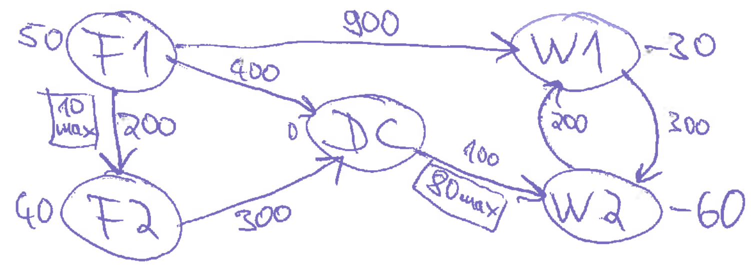
(3) (d) Suppose one of the entries in U needs to be slightly decreased. Which one should we choose in order not to change the costs? (You will receive points only if you justify your answer correctly, and not for guessing.)

(4) (e) Suppose the entry F1 in b is increased by one unit, and simultaneously one of the entries W1 or W2 in b is decreased by one unit. Which entry should be decreased in order to be most cost effective? What will be the extra cost for the increase/decrease?

(3) (f) Suppose the optimal flows in this problem can only be done with n units at the same time. What are the possible numbers n here? How and why can we find the biggest such number n in general, given the parameters b, C, U?

Problem 5: Extra Space

a)



b) Objective: Minimize cost $Z=\sum_{i,j} c_{ij}x_{ij}$.

Constraints: $\sum_j x_{ij} - \sum_j x_{ji} = b_i \quad \forall i \in N$

and $0 \le x_{ij} \le u_{ij}$ for all $(i,j) \in A$.

c) • c_{ij} = transportation cost from i to j

• u_{ij} = maximum capacity on arc (i,j)

• b_i = supply/demand or source/sink

↳ $b_i > 0$: supply/source
↳ $b_i < 0$: demand/sink
↳ $b_i = 0$: transshipment

d) Looking at the dual variables (shadow prices) in ln[10], decreasing $u_{F1,F2}$ by a unit will not change the cost.

e) According to shadow prices in ln[9]:

• increasing b_{F1} by one unit costs 700

• decreasing b_{W1} by one unit cost $-1 \cdot 200 = -200$

$\Rightarrow$ Should decrease b_{W1}, for a total extra cost of 500.

f) Due to the integer solution property, we can choose $n = 10, 5, 2, 1$ here. The biggest number n is the greatest common divisor of all b_i and u_{ij}.

In [1]:
```
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```
b = {'F1':50,
     'F2':40,
     'DC':0,
     'W1':-30,
     'W2':-60}

C = {('F1','F2'):200,
     ('F1','DC'):400,
     ('F1','W1'):900,
     ('F2','DC'):300,
     ('DC','W2'):100,
     ('W1','W2'):300,
     ('W2','W1'):200}

U = {('F1','F2'):10,
     ('DC','W2'):80}

N = list(b.keys())
A = list(C.keys())
V = list(U.keys())
```

In [3]:
```
model = ConcreteModel()
model.f = Var(A, within=NonNegativeReals)
```

In [4]:
```
def flow_rule(model, n):
    InFlow  = sum(model.f[i,j] for (i,j) in A if j==n)
    OutFlow = sum(model.f[i,j] for (i,j) in A if i==n)
    return InFlow + b[n] == OutFlow

model.flow = Constraint(N, rule=flow_rule)
```

In [5]:
```
def capacity_rule(model, i, j):
    return model.f[i,j] <= U[i,j]

model.capacity = Constraint(V, rule=capacity_rule)
```

In [6]:
```
model.cost = Objective(expr = sum(model.f[a]*C[a] for a in A), sense=minimize)
```

In [7]:
```
model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
model.f.get_values()
```

Out[7]:
```
{('F1', 'F2'): 0.0,
 ('F1', 'DC'): 40.0,
 ('F1', 'W1'): 10.0,
 ('F2', 'DC'): 40.0,
 ('DC', 'W2'): 80.0,
 ('W1', 'W2'): 0.0,
 ('W2', 'W1'): 20.0}
```

In [8]:
```
model.cost.expr()
```

Out[8]:
```
49000.0
```

In [9]:
```
for n in N:
    print(n, model.dual[model.flow[n]])
```

```
F1 700.0
F2 -600.0
DC -300.0
W1 200.0
W2 0.0
```

In [10]:
```
for j in V:
    print(j, model.dual[model.capacity[j]])
```

```
('F1', 'F2') 0.0
('DC', 'W2') -200.0
```