Jacobs University
Fall 2022

December 20, 2022

Operations Research

Final Exam

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

Name: Solutions

Matric. No.: ____________________

Problem 1: Graphical Method [25 points]
Consider the following Linear Programming problem: Maximize

$$Z = x_1 + 3x_2$$

subject to

$$\begin{aligned} -x_1 + 2x_2 &\le 6, \\ 3x_1 + 4x_2 &\le 36, \\ x_2 &\ge 3, \\ x_1, x_2 &\ge 0. \end{aligned}$$

(8) (a) Solve the problem with the graphical method, i.e., draw the feasible region, and find the optimal solution and the corresponding value of Z graphically.

(5) (b) From part (a), identify the two binding constraints. Using this information, compute the optimal solution and the corresponding value of Z.

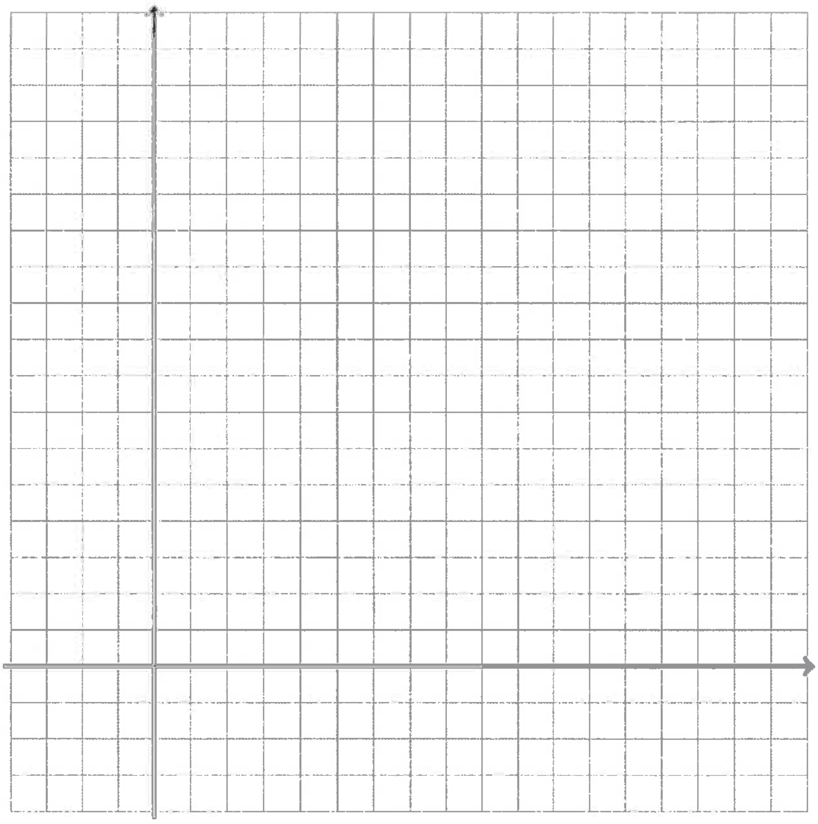
(2) (c) How many corner points does the feasible region have?

(5) (d) Write down the dual LP problem to the primal LP problem above. Without solving the dual problem explicitly, what is the value of $b^T y$, where $b^T = (6, 36, -3)$, and y is the optimal solution of the dual problem?

(5) (e) Find out which one of the following is the vector of shadow prices for this problem. Explain your answer! (Hint: You can figure out the answer directly from parts (b) and (d).)

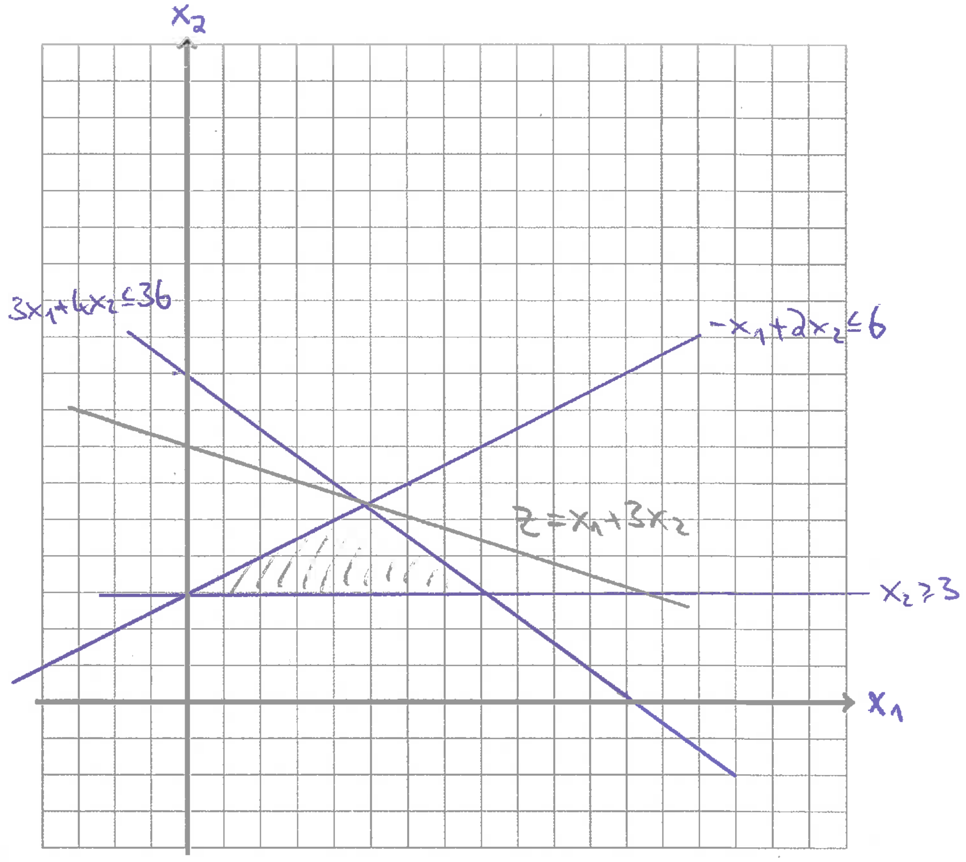
- $(y_1, y_2, y_3) = (\frac{1}{2}, 0, \frac{1}{2})$,
- $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$,
- $(y_1, y_2, y_3) = (\frac{1}{2}, \frac{1}{2}, 0)$,
- $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{3}{2})$,
- $(y_1, y_2, y_3) = (\frac{3}{2}, \frac{1}{2}, 0)$.

Problem 1: Extra Space



Problem 1: Extra Space

a)



$\Rightarrow$ The optimal solution is $x_1 \approx 4.8$, $x_2 \approx 5.4$, with $z \approx 21$

b) The first two constraints are binding, i.e., to find the optimal solution we need to solve

$$-x_1 + 2x_2 = 6 \qquad 3R1+R2 \Rightarrow 10x_2 = 54 \Rightarrow x_2 = 5.4$$
$$3x_1 + 4x_2 = 36$$
$$\Rightarrow x_1 = 10.8 - 6 = 4.8$$

$\Rightarrow z = x_1 + 3x_2 = 4.8 + 16.2 = 21$

c) The feasible region is a triangle, i.e., there are 3 corner points.

d) In general, given a primal problem: • maximize $c^T x$,
• subject to $Ax \leq b$ and $x \geq 0$,

the dual problem is: • minimize $b^T y$,
• subject to $A^T y \geq c$ and $y \geq 0$.

Problem 1: Extra Space

$\Rightarrow$ Here, the dual problem is: minimize $6y_1 + 36y_2 - 3y_3$,

subject to

$$-y_1 + 3y_2 \geq 1,$$
$$2y_1 + 4y_2 - y_3 \geq 3,$$
$$y_1, y_2, y_3 \geq 0.$$

By strong duality, $b^T y = c^T x$, at the optimal solutions, i.e., here $b^T y = 21$.

e) The first two constraints are binding (part (b)), thus $y_3 = 0$ (non-binding).

Furthermore: $\underbrace{\begin{pmatrix} 6 \\ 36 \\ -3 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}}_{b^T y} = 3 + 18 = 21$, while $\begin{pmatrix} 6 \\ 36 \\ -3 \end{pmatrix} \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix} = 9 + 18 = 27$.

Thus, using (d), the correct shadow price must be $(\frac{1}{2}, \frac{1}{2}, 0)$.

Problem 2: Standard Form and Simplex Method [25 points]

(3) (a) Consider the following Linear Programming problem: Maximize

$$Z = x_1 + 2x_2$$

subject to

$$3x_1 + x_2 \leq 21,$$
$$x_1 + 3x_2 \leq 15,$$
$$x_1, x_2 \geq 0,$$

Write this problem in standard form, i.e., as the problem to minimize

$$\tilde{Z} = c^{\mathrm{T}}\tilde{x}$$

subject to

$$A\tilde{x} = b, \quad \tilde{x} \geq 0.$$

Specify exactly the vectors b, c and the matrix A.

(20) (b) Solve the problem with the simplex method as shown in class. *(Note: You will not receive points for simply stating the solution, but only for using the simplex method step by step.)*

a) Standard form: • Minimize $\tilde{Z} = -x_1 - 2x_2$

• subject to $3x_1 + x_2 + s_1 = 21$,

$x_1 + 3x_2 + s_2 = 15$,

$x_1, x_2, s_1, s_2 \geq 0$.

$\Rightarrow c = (-1, -2, 0, 0)^T$,

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}, \quad \tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}, \quad b = \begin{pmatrix} 21 \\ 15 \end{pmatrix}$$

b) The simplex tableau is

3	1	1	0	21
1	3	0	1	15
-1	-2	0	0	0

$\Rightarrow$ R1 − R2/3 → R1

R2/3 → R2

$\frac{2}{3}$R2 + R3 → R3

$\frac{8}{3}$	0	1	$-\frac{1}{3}$	16
$\frac{1}{3}$	1	0	$\frac{1}{3}$	5
$-\frac{1}{3}$	0	0	$\frac{2}{3}$	10

Problem 2: Extra Space

$$\begin{array}{l} \frac{3}{8}R1 \to R1 \\ -\frac{1}{8}R1 + R2 \to R2 \\ \frac{1}{8}R1 + R3 \to R3 \end{array} \qquad \left[\begin{array}{cccc|c} 1 & 0 & \frac{3}{8} & -\frac{1}{8} & 6 \\ 0 & 1 & -\frac{1}{8} & \frac{3}{8} & 3 \\ \hline 0 & 0 & \frac{1}{8} & \frac{5}{8} & 12 \end{array}\right]$$

$\Rightarrow$ The optimal solution is $(x_1, x_2, s_1, s_2) = (6, 3, 0, 0)$, with

$z = 12 = -\tilde{z}$.

Problem 2: Extra Space

Problem 2: Extra Space

Problem 3: Inventory Theory [25 points]
We consider the following periodic review problem (which we also discussed in class):

- There are four production/delivery periods $i = 1, 2, 3, 4$.
- The demands in the four periods are $r_1 = 3$, $r_2 = 2$, $r_3 = 3$, $r_4 = 2$, and they have to be met exactly.
- The setup costs for starting a production is $K = 2$ (million Dollars).
- The holding costs for each unit left in inventory at the end of the period is $h = 0.2$ (million Dollars).

The goal is to find the production schedule that minimizes the costs.

Note that the production cost per item is neglected here because it is the same in any production schedule. Also note that production should only be started if the inventory is empty (otherwise the production schedule is not optimal).

Find the optimal production schedule and total production cost *using dynamic programming.*

(Hint: Use the state s = number of items in inventory, and the decision variables x_i = number of items produced in period i. Then set up the correct cost function $f_i(s, x_i)$.)

The cost function is $f_i(s,x_i) = \begin{cases} hs + f_{i+1}^*(s-r_i+x_i) + K, & s=0 \leftarrow \text{produce} \\ hs + f_{i+1}^*(s-r_i+x_i), & s>0 \leftarrow \text{don't produce} \end{cases}$

With dynamic programming we find:

$i=4$: $f_4^*(s) = \begin{cases} K = 2 & \text{if } s=0 \\ hs = 0.2\cdot 2 = 0.4 & \text{if } s = 2 = r_4 \end{cases}$

$i=3$:

	$f_3(s,x_3) = hs + f_4^*(s-\overset{3}{r_3}+x_3)$ $(+K \text{ if } s=0)$				
	$x_3=0$	$x_3=3$	$x_3=5$	f_3^*	x_3^*
$s=0$	/	$2+2=4$	$0.4+2=2.4$	2.4	5
$s=3$	$0.2\cdot 3+2=2.6$	/	/	2.6	0
$s=5$	$0.2\cdot 5+0.4=1.4$	/	/	1.4	0

Problem 3: Extra Space

$i=2$:

$$f_2(s,x_2)=hs+f_3^*(s-\overset{2}{\overset{\shortparallel}{r_2}}+x_2)\ (+K \text{ if } s=0)$$

	$x_2=0$	$x_2=2$	$x_2=5$	$x_2=7$	f_2^*	x_2^*
$s=0$	/	$2.4+2=4.4$	$2.6+2=4.6$	$1.4+2=3.4$	3.4	7
$s=2$	$0.2\cdot 2+2.4=2.8$	/	/	/	2.8	0
$s=5$	$0.2\cdot 5+2.6=3.6$	/	/	/	3.6	0
$s=7$	$0.2\cdot 7+1.4=2.8$	/	/	/	2.8	0

$i=1$:

$$f_1(s,x_1)=hs+f_2^*(s-\overset{3}{\overset{\shortparallel}{r_1}}+x_1)\ (+K \text{ if } s=0)$$

	$x_1=3$	$x_1=5$	$x_1=8$	$x_1=10$	f_1^*	x_1^*
$s=0$	$3.4+2=5.4$	$2.8+2=4.8$	$3.6+2=5.6$	$2.8+2=4.8$	4.8	5 or 10

There are 2 optimal production schedules:

- produce 10 items in period 1, or
- produce 5 items in period 1, and 5 items in period 3.

Both lead to an optimal production cost of 4.8 (million Dollars)

Problem 3: Extra Space

Problem 3: Extra Space

Problem 4: Decision Analysis [25 points]
Consider the decision analysis problem with the following payoff table.

Alternative	Payoff in State S_i	
	S_1	S_2
A_1	700	-100
A_2	90	90
Prior Probability	1/4	3/4

(8) (a) Compute the expected payoff (with the prior probabilities) for each alternative. Given the result, which alternative should be chosen?

(9) (b) Next, some experimentation is done in order to improve the decision making. The outcome of the experimentation can be positive ("pos") or negative ("neg"). The probabilities that the outcome is positive/negative given state S_1/S_2 are given by

$$P(\text{pos} \mid S_1) = 0.6,$$
$$P(\text{pos} \mid S_2) = 0.2,$$
$$P(\text{neg} \mid S_1) = 0.4,$$
$$P(\text{neg} \mid S_2) = 0.8.$$

(4) (i) Compute the total probabilities $P(\text{pos})$ and $P(\text{neg})$ for the outcome to be positive or negative. *(Note: The result for negative is $P(neg) = 0.7$; points for this exercise will be given for the correct computation.)*

(5) (ii) Use Bayes' rule to compute the probabilities that state S_1 is realized, given a positive/negative outcome of the experimentation, i.e., compute $P(S_1 \mid \text{pos})$ and $P(S_1 \mid \text{neg})$. *(Note: The results are $P(S_1 \mid pos) = 0.5$ and $P(S_1 \mid neg) = 1/7$; points for this exercise will be given for the correct computation.)*

(8) (c) Given the results of part (b), set up the decision tree for this problem. The expected payoffs for each step do *not* need to be computed! The tree needs to include the possibilities "with/without experimentation", "positive/negative outcome", "alternative A_1/A_2", "state S_1/S_2". The tree should include a variable c for the cost of experimentation, the necessary probabilities, and the total payoffs at the end only (no intermediate payoffs need to be computed).

Problem 4: Extra Space

Problem 4: Extra Space

a) $\mathbb{E}(A_1) = 700 \cdot \frac{1}{4} + (-100) \cdot \frac{3}{4} = 100$

$\mathbb{E}(A_2) = 90 \cdot \frac{1}{4} + 90 \cdot \frac{3}{4} = 90$

Given this result, A_1 seems preferable.

b) (i) $\mathbb{P}(pos) = \mathbb{P}(pos|S_1)\mathbb{P}(S_1) + \mathbb{P}(pos|S_2)\mathbb{P}(S_2)$

$= 0.6 \cdot \frac{1}{4} + 0.2 \cdot \frac{3}{4} = \frac{6}{40} + \frac{6}{40} = \frac{12}{40} = 0.3$

$\Rightarrow \mathbb{P}(neg) = 1 - \mathbb{P}(pos) = 0.7$

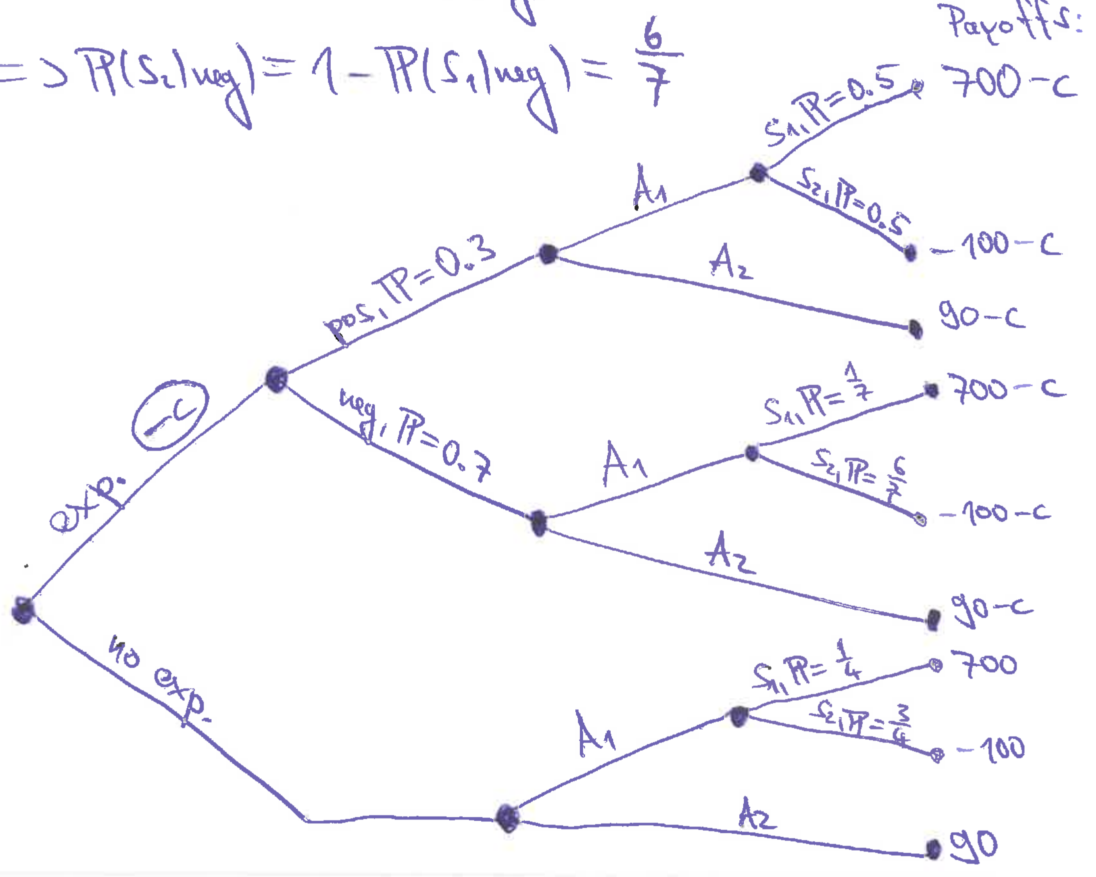
(ii) $\mathbb{P}(S_1|pos) = \frac{\mathbb{P}(pos|S_1) \cdot \mathbb{P}(S_1)}{\mathbb{P}(pos)} = \frac{0.6 \cdot \frac{1}{4}}{0.3} = \frac{2}{4} = 0.5$

$\Rightarrow \mathbb{P}(S_2|pos) = 1 - \mathbb{P}(S_1|pos) = 0.5$

$\mathbb{P}(S_1|neg) = \frac{\mathbb{P}(neg|S_1)\mathbb{P}(S_1)}{\mathbb{P}(neg)} = \frac{0.4 \cdot \frac{1}{4}}{0.7} = \frac{1}{7}$

$\Rightarrow \mathbb{P}(S_2|neg) = 1 - \mathbb{P}(S_1|neg) = \frac{6}{7}$

c)



Problem 4: Extra Space

Problem 5: Pyomo [25 points]
The pyomo program on the last exam page shows an implementation of a minimum cost flow problem.

(6) (a) Draw the corresponding network representation of the problem. (Make sure to include the given values of the parameters from the pyomo program.)

(5) (b) Write down the Linear Programming problem that is solved in mathematical notation (i.e., write down the objective function and the constraints).

(5) (c) Briefly explain the meaning of the parameters b, C, U in the context of minimum cost flow problems. In particular, explain what positive, negative, or zero values for entries in b mean.

(3) (d) Suppose one of the entries in U can be slightly increased. Which one should we choose in order to lower the costs? (You will receive points only if you justify your answer correctly, and not for guessing.)

(3) (e) Suppose one of the entries F1, F2, or F3 in b is increased by one unit, and simultaneously one of the entries W1, W2, or W3 in b is decreased by one unit. Which entries should be increased/decreased in order to be most cost effective? What will be the extra cost for the increase/decrease?

(3) (f) Management notices that there is some flexibility between F1, F2, and F3 concerning the units of b, i.e., one of them could be increased by a few units while simultaneously decreasing another one by the same number of units. Which of the following four solutions should we recommend in order to be most cost effective:

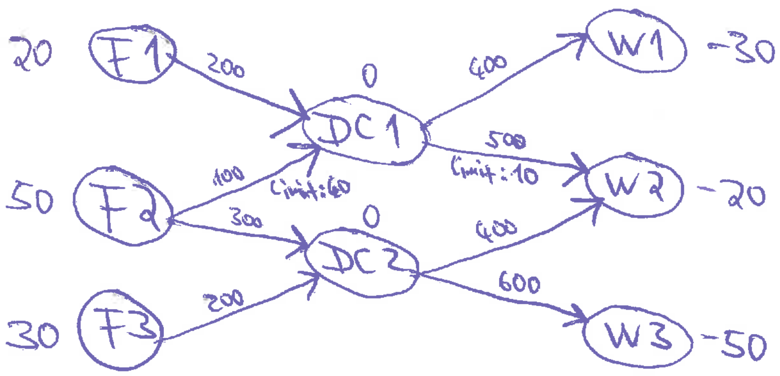
1. Increase F1 by a few units and decrease F2.
2. Increase F2 by a few units and decrease F3.
3. Increase F3 by a few units and decrease F1.
4. Increase F3 by a few units and decrease F2.

Explain your answer. (You will not receive points for a random choice.)

Problem 5: Extra Space

Problem 5: Extra Space

a)



b) Minimize $Z = \sum_{(i,j)\in A} c_{ij} x_{ij}$,

subject to $\sum_j x_{ij} - \sum_j x_{ji} = b_i$ and $0 \le x_{ij} \le u_{ij}$ $\forall (i,j) \in A$.

c) b_i = supply/demand at node i

↳ $b_i > 0 \Rightarrow$ supply/source node

↳ $b_i < 0 \Rightarrow$ demand/sink node

↳ $b_i = 0 \Rightarrow$ transshipment node

c_{ij} = transportation cost from i to j

u_{ij} = maximum capacity on arc (i,j)

d) The entry with the smallest shadow price should be chosen.

Here, U(DC1,W2) has smallest shadow price −100, i.e., this should be chosen.

e) We choose the combination with the smallest shadow price $y_{F_i} - y_{W_i}$.

Here, F3 should be increased, and W1 decreased, for a total cost of −100 − (−500) = 400.

f) We should increase F3 and decrease F1, because per unit this will lower the cost by −100 − 100 = −200 according to their shadow prices, which is the biggest decrease. ⇒ Choose option 3.

Problem 5: Extra Space

In [1]:
```python
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```python
b = {'F1':20,
     'F2':50,
     'F3':30,
     'DC1':0,
     'DC2':0,
     'W1':-30,
     'W2':-20,
     'W3':-50}

C = {('F1','DC1'):200,
     ('F2','DC1'):100,
     ('F2','DC2'):300,
     ('F3','DC2'):200,
     ('DC1','W1'):400,
     ('DC1','W2'):500,
     ('DC2','W2'):400,
     ('DC2','W3'):600}

U = {('F2','DC1'):40,
     ('DC1','W2'):10}

N = list(b.keys())
A = list(C.keys())
V = list(U.keys())
```

In [3]:
```python
model = ConcreteModel()
model.x = Var(A, within=NonNegativeReals)
```

In [4]:
```python
def flow_rule(model, n):
    InFlow  = sum(model.x[i,j] for (i,j) in A if j==n)
    OutFlow = sum(model.x[i,j] for (i,j) in A if i==n)
    return OutFlow - InFlow == b[n]

model.flow = Constraint(N, rule=flow_rule)
```

In [5]:
```python
def capacity_rule(model, i, j):
    return model.x[i,j] <= U[i,j]

model.capacity = Constraint(V, rule=capacity_rule)
```

In [6]:
```python
model.cost = Objective(expr = sum(model.x[a]*C[a] for a in A), sense=minimize)
```

In [7]:
```python
model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
model.x.get_values()
```

Out[7]:
```
{('F1', 'DC1'): 20.0,
 ('F2', 'DC1'): 20.0,
 ('F2', 'DC2'): 30.0,
 ('F3', 'DC2'): 30.0,
 ('DC1', 'W1'): 30.0,
 ('DC1', 'W2'): 10.0,
 ('DC2', 'W2'): 10.0,
 ('DC2', 'W3'): 50.0}
```

In [8]:
```python
model.cost.expr()
```

Out[8]:
```
72000.0
```

In [9]:
```python
for n in N:
    print(n, model.dual[model.flow[n]])
```

```
F1 100.0
F2 0.0
F3 -100.0
DC1 -100.0
DC2 -300.0
W1 -500.0
W2 -700.0
W3 -900.0
```

In [10]:
```python
for j in V:
    print(j, model.dual[model.capacity[j]])
```

```
('F2', 'DC1') 0.0
('DC1', 'W2') -100.0
```