Jacobs University
Fall 2022

November 1, 2022

Operations Research

Midterm Exam

(For bonus points only.)

Instructions:

- Do all the work on this exam paper.
- Show your work, i.e., carefully write down the steps of your solution. You will receive points not just based on your final answer, but on the correct steps in your solution.
- No tools or other resources are allowed for this exam. In particular, no notes and no calculators.

Name: Solutions

Matric. No.: ______________________

Problem 1: Graphical Method [25 points]
Consider the following Linear Programming problem: Maximize

$$Z = x_1 + 3x_2$$

subject to

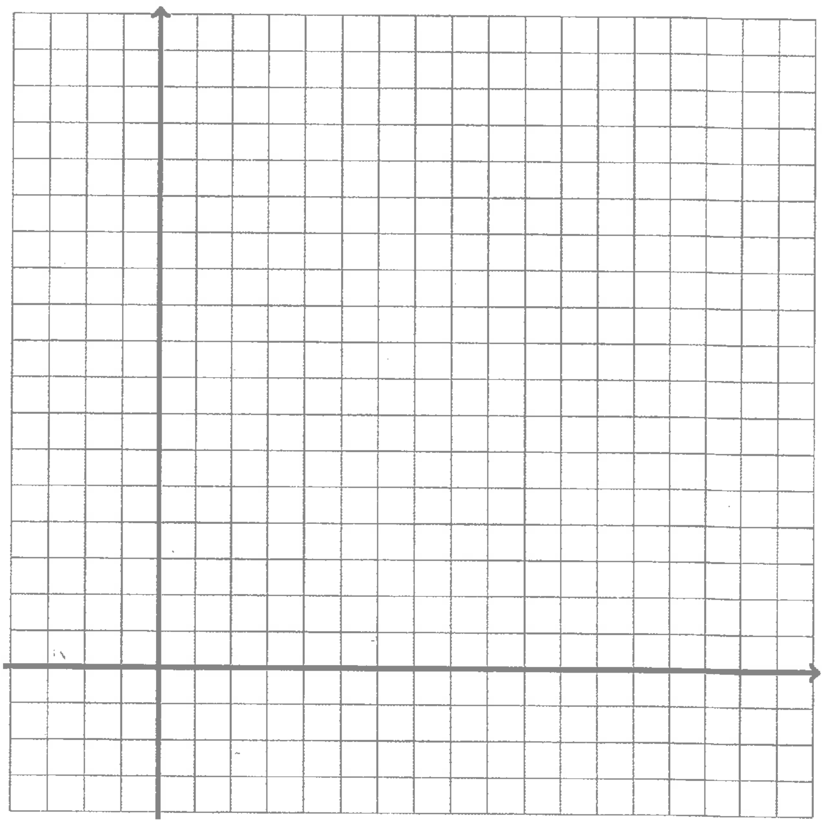
$$-x_1 + 3x_2 \leq 15, \quad \text{(i)}$$
$$\frac{4}{3}x_1 + x_2 \leq 12, \quad \text{(ii)}$$
$$2x_1 + 3x_2 \leq 24, \quad \text{(iii)}$$
$$x_1, x_2 \geq 0.$$

(9) (a) Solve the problem with the graphical method, i.e., draw the feasible region, and find the optimal solution and the corresponding value of Z.

(6) (b) From part (a), identify the two binding constraints. Using this information, compute the optimal solution and the corresponding value of Z.

(4) (c) Give one example of an objective function for this problem (i.e., keeping the constraints) that leads to infinitely many optimal solutions.

(6) (d) Write down the dual LP problem to the primal LP problem above. Without solving the dual problem explicitly, what is the value of $b^T y$, where $b^T = (15, 12, 24)$, and y is the optimal solution of the dual problem?



Problem 1: Extra Space

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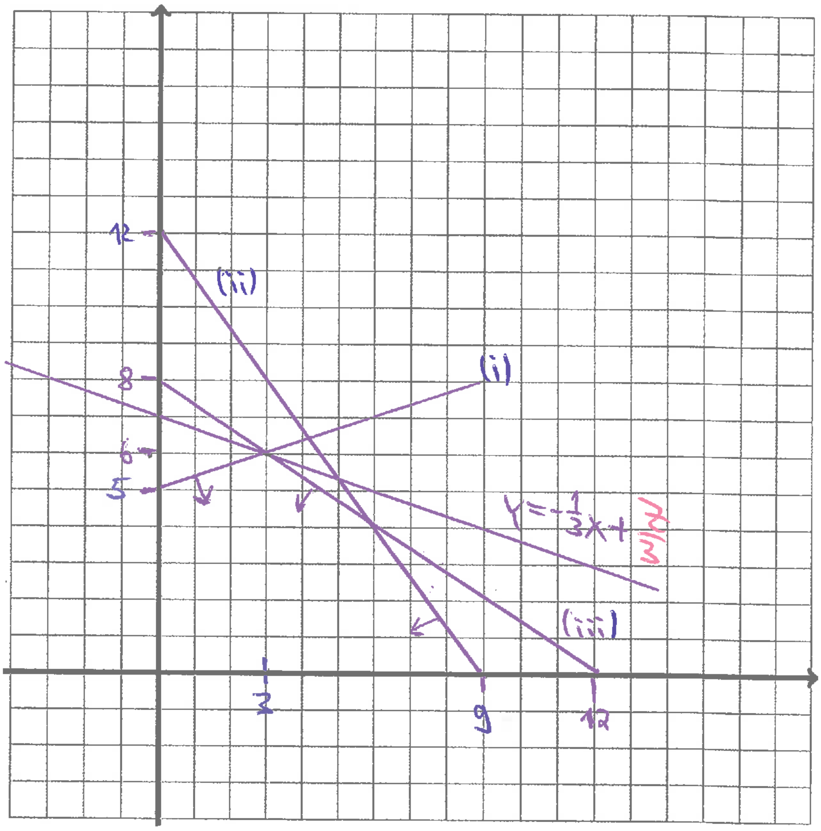
a)



optimal solution: $x_1 = 3$, $x_2 = 6$ with $z = 21$

b) Binding constraints: $-x_1 + 3x_2 \leq 15$ and $2x_1 + 3x_2 \leq 24$

$\Rightarrow$ Need to solve:

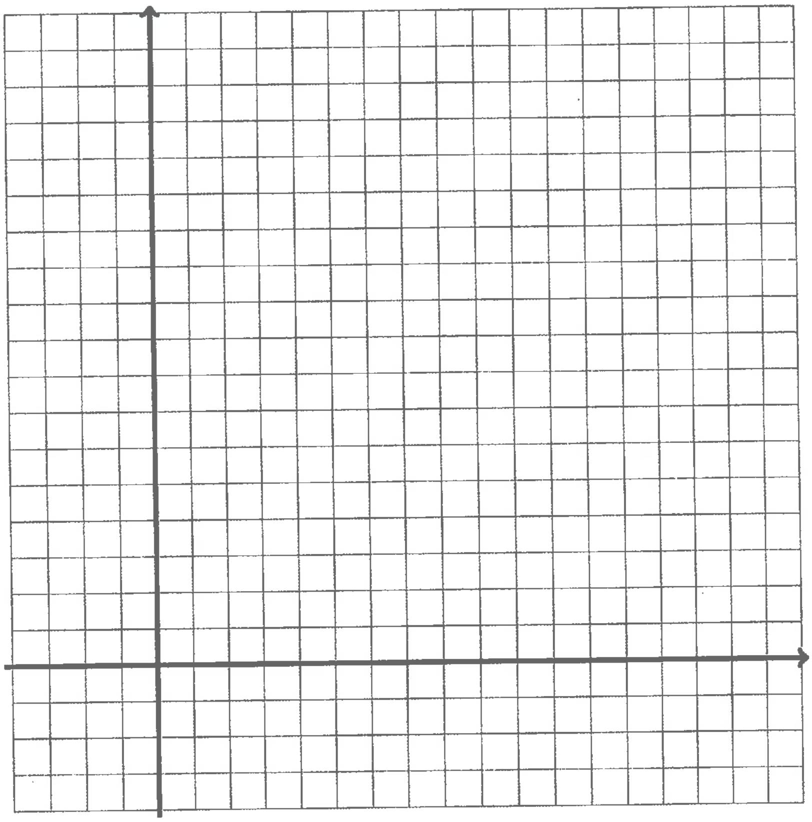
$$-x_1 + 3x_2 = 15 \quad (1)$$
$$2x_1 + 3x_2 = 24 \quad (2)$$

$(2) - (1) \Rightarrow 3x_1 = 9 \Rightarrow x_1 = 3$ and thus $-3 + 3x_2 = 15$, i.e., $x_2 = 6$.

$\Rightarrow z = 3 + 3 \cdot 6 = 21$

c) E.g., $z = 2x + 3y$ leads to ∞ many solutions, namely any point in the line segment between $(3,6)$ and $(6,4)$.
This is because the slope of this objective fct. is $-\frac{2}{3}$, just like the third constraint.

Problem 1: Extra Space



d) In general, if "max. $z = c^T x$ subject to $Ax \leq b$ and $x \geq 0$" is the primal problem, then the dual problem is:

Minimize $b^T y$, subject to $A^T y \geq c$ and $y \geq 0$.

Here, the dual problem is:

Minimize $(15, 12, 24) \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 15y_1 + 12y_2 + 24y_3$,

subject to $\begin{pmatrix} -1 & \frac{4}{3} & 2 \\ 3 & 1 & 3 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $y_1, y_2, y_3 \geq 0$.

By strong duality, $b^T y = c^T x = 21$ here.

Problem 2: Standard Form and Simplex Method [25 points]

(5) (a) Consider the following Linear Programming problem: Maximize

$$Z = x_1 + 2x_2$$

subject to

$$3x_1 + x_2 \leq 21,$$
$$x_1 + 3x_2 \leq 15,$$
$$x_1, x_2 \geq 0,$$

Write this problem in standard form, i.e., as the problem to minimize

$$\tilde{Z} = c^{\mathrm{T}} \tilde{x}$$

subject to

$$A\tilde{x} = b, \quad \tilde{x} \geq 0.$$

Specify exactly the vectors b, c and the matrix A.

(20) (b) Solve the problem with the simplex method as shown in class. *(Note: You will not receive points for simply stating the solution, but only for using the simplex method step by step.)*

a) Standard form: Minimize $\tilde{z} = c^T \tilde{x}$, subject to $A\tilde{x} = b$ and $\tilde{x} \geq 0$.

Here, $\tilde{x} = \begin{pmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{pmatrix}$, $c = \begin{pmatrix} -1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 21 \\ 15 \end{pmatrix}$, and $A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}$.

b) We set up a simplex tableau as in class:

3	1	1	0	21
1	3	0	1	15
-1	-2	0	0	0

=>

$R1 - \frac{R2}{3}$	$\frac{8}{3}$	0	1	$-\frac{1}{3}$	16
$\frac{R2}{3}$	$\frac{1}{3}$	1	0	$\frac{1}{3}$	5
$\frac{2}{3}R2 + R3$	$-\frac{1}{3}$	0	0	$\frac{2}{3}$	10

=>

$\frac{3}{8} \cdot R1$	1	0	$\frac{3}{8}$	$-\frac{1}{8}$	6
$-\frac{R1}{8} + R2$	0	1	$-\frac{1}{8}$	$\frac{3}{8}$	3
$\frac{R1}{8} + R3$	0	0	$\frac{1}{8}$	$\frac{5}{8}$	12

=> optimal solution is $(x_1, x_2) = (6, 3)$ with $z = 12$

Problem 2: Extra Space

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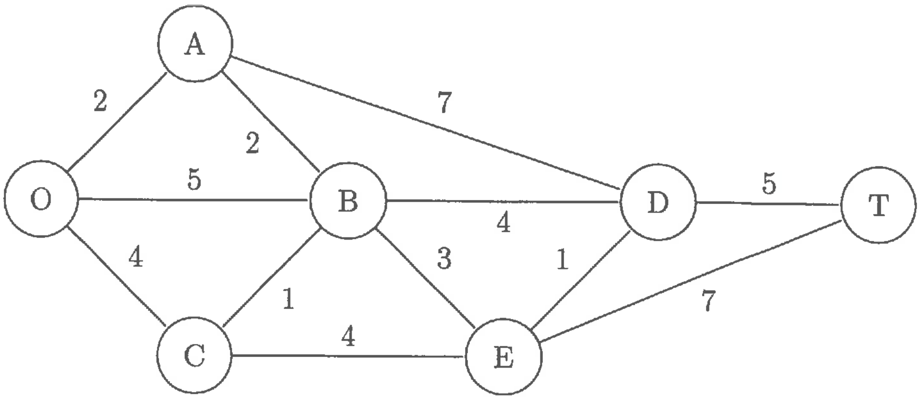
Problem 3: Network optimization [25 points]
Consider the following network (the Seervada Park problem that we discussed in class):



Find the shortest path from the origin "O" to the target "T" using the shortest path algorithm discussed in class. *(Note: You will not receive points for simply stating the solution, but only for using the shortest path method step by step.)*

See Sessions 14 and 15 lecture notes.

Problem 3: Extra Space

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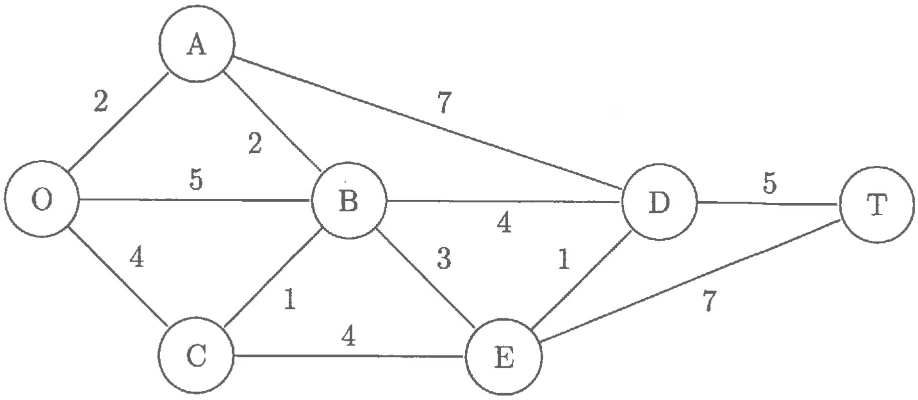


Problem 3: Extra Space

Problem 4: Pyomo [25 points]
The pyomo program on the last page shows an implementation of a transportation problem.

5 (a) Draw a network representation of the problem.

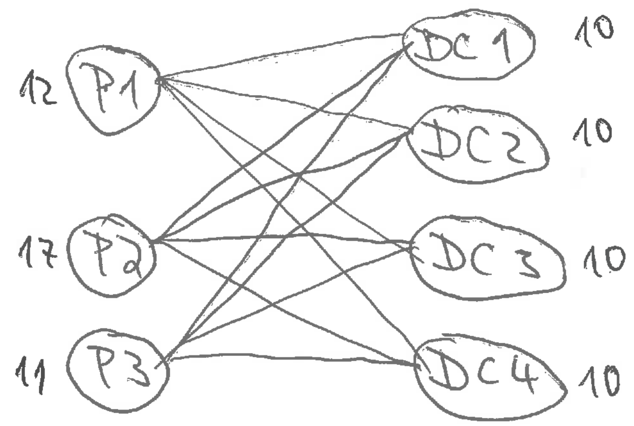
5 (b) Write down the corresponding Linear Programming problem in mathematical notation.

5 (c) Briefly explain the meaning of the parameters P, DC, d, s, c in the context of transportation problems.

5 (d) Suppose the supply in one of the entries of s and the demand in one of the entries of d is increased by 1 unit. Which entry should we choose for the supply increase to be most cost effective? To which entry in d should we ideally deliver the extra supply (to be most cost effective)? What will be the new optimal total cost?

5 (e) Now suppose the demand in the first entry of d goes up from 10 to 12 units, but the supply stays the same. This means the DC's will be undersupplied by 2 units in total. How do we need to change the code in order to solve this problem? (Explicitly write down the new code here with explanations.)

a)



b) Minimize $z = \sum_{i,j} c_{ij} x_{ij}$, subject to $\sum_j x_{ij} \le s_i$ and $\sum_i x_{ij} \ge d_j$, and $x_{ij} \ge 0$.

c) P = production facilities
DC = distribution centers
d_j = demand at DC j
s_i = supply at P i
c_{ij} = transportation cost from i to j

Problem 4: Extra Space

d) To answer this question we look at the shadow prices in In[8] and In[9]. Increasing one unit in DC 3, and in either P1 or P3 increases the cost the least, namely by 600 − 200 = 400. Thus, the new optimal cost will be 32 800.

e) In In[2] we need to replace:

- P by P = [1,2,3,4] ; Here P4 is a dummy supply;
- d by d = {1:12, 2:10, 3:10, 4:10} to accound for the new demand;
- s by s = {1:12, 2:17, 3:11, 4: 2} since the dummy supply at P4 is 2;
- In c we need to add an extra line
  (4,1):0, (4:2):0, (4:3):0, (4:4):0 to set the cost for shipping from the dummy supply equal to 0.

Problem 4: Extra Space

Problem 4: Extra Space

In [1]:
```python
from pyomo.environ import *
from pyomo.opt import *
opt = solvers.SolverFactory("glpk")
```

In [2]:
```python
P = [1, 2, 3]
DC = [1, 2, 3, 4]
d = {1:10, 2:10, 3:10, 4:10}
s = {1:12, 2:17, 3:11}
c = {(1,1):800, (1,2):1300, (1,3):400, (1,4):700,
(2,1):1100, (2,2):1400, (2,3):600, (2,4):1000,
(3,1):600, (3,2):1200, (3,3):800, (3,4):900}

model = ConcreteModel()
```

In [3]:
```python
model.x = Var(P, DC, within=NonNegativeReals)
model.z = Objective(expr = sum(c[i,j]*model.x[i,j] for i in P for j in DC), sense=minimize)
```

In [4]:
```python
def supply_rule (model, i):
    return sum(model.x[i,j] for j in DC) <= s[i]
model.supply = Constraint(P, rule=supply_rule)

def demand_rule (model, j):
    return sum(model.x[i,j] for i in P) >= d[j]
model.demand = Constraint(DC, rule=demand_rule)
```

In [5]:
```python
model.dual = Suffix(direction=Suffix.IMPORT)
results = opt.solve(model)
```

In [6]:
```python
model.x.get_values()
```

Out[6]:
```
{(1, 1): 0.0,
 (1, 2): 0.0,
 (1, 3): 2.0,
 (1, 4): 10.0,
 (2, 1): 0.0,
 (2, 2): 9.0,
 (2, 3): 8.0,
 (2, 4): 0.0,
 (3, 1): 10.0,
 (3, 2): 1.0,
 (3, 3): 0.0,
 (3, 4): 0.0}
```

In [7]:
```python
model.z.expr()
```

Out[7]:
```
32400.0
```

In [8]:
```python
for j in DC:
    print(j, model.dual[model.demand[j]])
```

```
1 800.0
2 1400.0
3 600.0
4 900.0
```

In [9]:
```python
for i in P:
    print(i, model.dual[model.supply[i]])
```

```
1 -200.0
2 0.0
3 -200.0
```