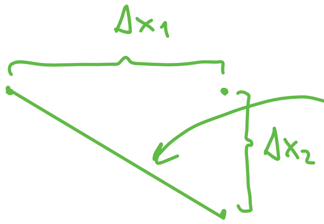
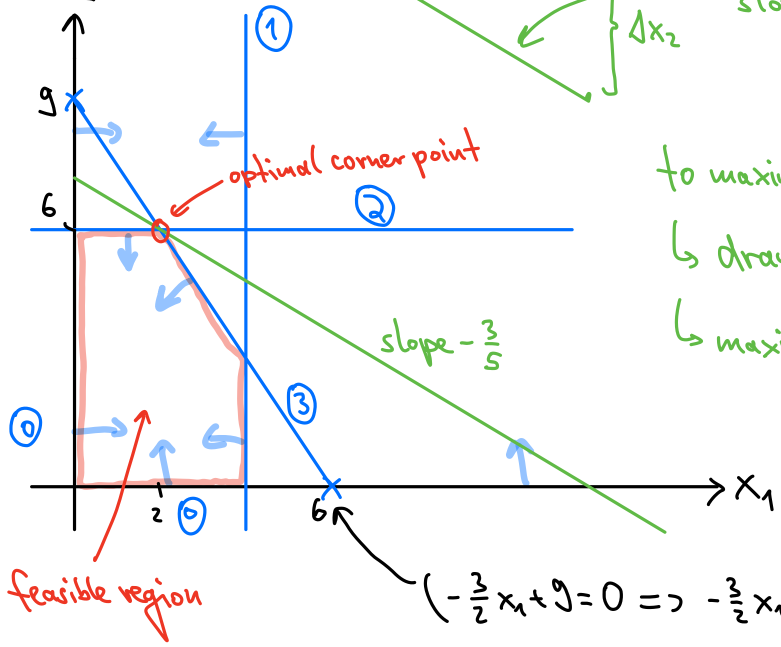
Recall that last time we discussed the following problem:

- Maximize $Z = 3x_1 + 5x_2$ $\longrightarrow$ $5x_2 = Z - 3x_1 \Rightarrow x_2 = \underbrace{-\frac{3}{5}}_{\text{slope}} x_1 + \underbrace{\frac{Z}{5}}_{\text{intercept with } x_2 \text{ axis}}$
- Constraints:
  - $x_1, x_2 \geq 0$ (0)
  - $x_1 \leq 4$ (1)
  - $2x_2 \leq 12$ (2) $(x_2 \leq 6)$
  - $3x_1 + 2x_2 \leq 18$ (3) $\longrightarrow$ solve for x_2: $2x_2 \leq -3x_1 + 18$
    $\Rightarrow x_2 \leq -\frac{3}{2}x_1 + 9$

4. Graphical solution:



slope $m = \frac{\Delta x_2}{\Delta x_1} = \frac{-3}{5}$



to maximize: $Z = 3x_1 + 5x_2$

↳ draw $x_2 = \underbrace{\frac{1}{5}Z} - \frac{3}{5}x_1 \Rightarrow$ line with slope $-\frac{3}{5}$

↳ maximize intercept with x_2 axis

$\left(-\frac{3}{2}x_1 + 9 = 0 \Rightarrow -\frac{3}{2}x_1 = -9 \Rightarrow x_1 = 9 \cdot \frac{2}{3} = 6\right)$

optimal corner point:
- read off: $x_1 = 2,\ x_2 = 6$
- or: (2): $2x_2 = 12 \Rightarrow x_2 = 6$

  (3): $3x_1 + \underbrace{2x_2}_{=6} = 18 \Rightarrow 3x_1 = 6 \Rightarrow x_1 = 2$

6. Recommendation:

↳ produce 2 batches of product 1, and 6 batches of product 2

↳ then profit will be maximal, namely $z = 3x_1 + 5x_2 = 3\cdot 2 + 5\cdot 6 = 36$ (k\$)

2. Linear Programming

2.1 Graphical Solutions

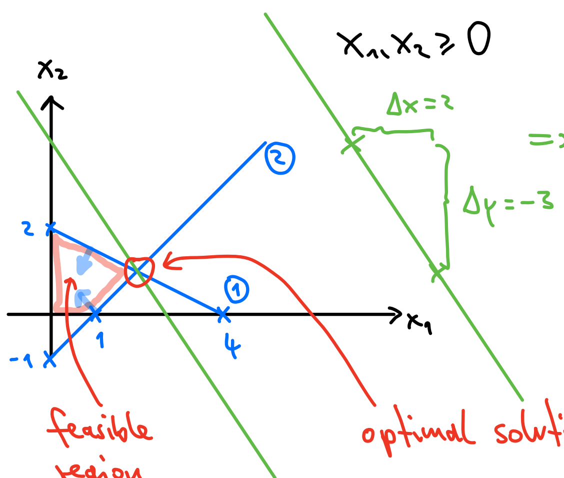
We consider examples to illustrate different possibilities for solutions.

1. Similar to introductory example:

- maximize $z = 3x_1 + 2x_2$ ($\Rightarrow$ line $x_2 = -\frac{3}{2}x_1 + \frac{z}{2}$, i.e., slope $-\frac{3}{2}$)
- with constraints $x_1 + 2x_2 \leq 4$ ① (note: ① $\Leftrightarrow x_2 \leq 2 - \frac{x_1}{2}$, so everything under the line is allowed)

  $x_1 - x_2 \leq 1$ ② (② $\Leftrightarrow x_2 \geq x_1 - 1$, so everything above the line is allowed)

  $x_1, x_2 \geq 0$



$\Rightarrow$ slope $= \frac{\Delta x_2}{\Delta x_1} = \frac{-3}{2}$ here

feasible region

optimal solution: a "cornerpoint feasible (CPF) solution"

At the $\boxed{\text{optimal corner point}}$ (where blue lines meet): $x_1 + 2x_2 = 4$, $x_1 - x_2 = 1$

$\Rightarrow$ augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 1 & -1 & 1 \end{array}\right)$

Gaussian elimination: $R1 - R2 \to R2$ $\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 3 & 3 \end{array}\right) \xrightarrow{R2/3} \left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R1-2R2 \to R1} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right)$

(more next time)

$\Rightarrow$ solution is $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, there $z = 3\cdot 2 + 2\cdot 1 = 8$.

↳ compare with picture above

$\Leftrightarrow x_2 = -\frac{2}{3}x_1 + \frac{z}{9}$

2. • minimize $z = 6x_1 + 9x_2$ (slope $-\frac{2}{3}$)

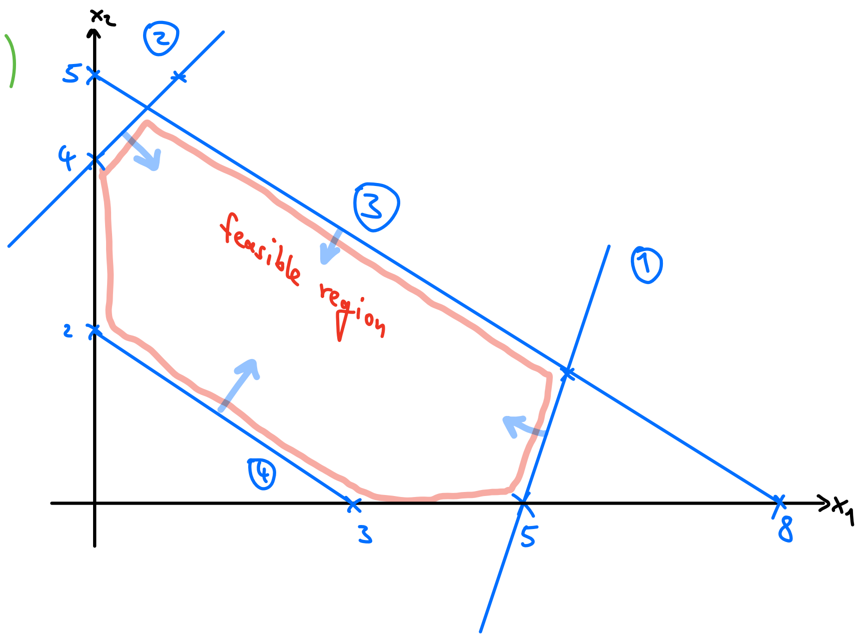
   • constraints: $3x_1 - x_2 \leq 15$ ①

   $-x_1 + x_2 \leq 4$ ②

   $5x_1 + 8x_2 \leq 40$ ③

   $2x_1 + 3x_2 \geq 6$ ④

   $x_1, x_2 \geq 0$



We continue this example next time.