Operations Research, Jacobs University, Fall 2022 Session 3

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We continue the examples from last time.

$\Longleftrightarrow x_2 = -\frac{2}{3}x_1 + \frac{z}{9}$

2. • minimize $z = 6x_1 + 9x_2$ (slope $-\frac{2}{3}$)

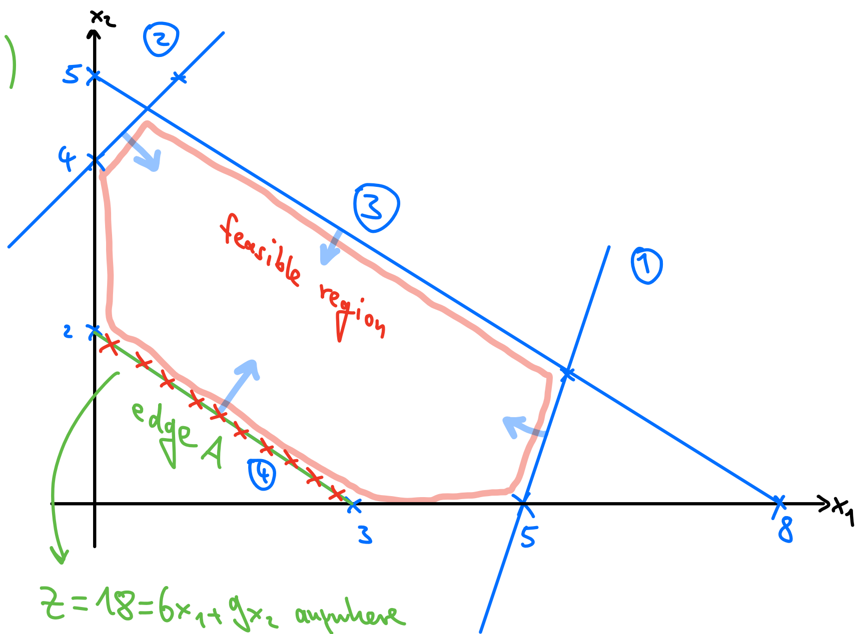
   • constraints: $3x_1 - x_2 \leq 15$ ①

   $-x_1 + x_2 \leq 4$ ②

   $5x_1 + 8x_2 \leq 40$ ③

   $x_2 \geq -\frac{2}{3}x_1 + 2 \Longleftarrow$ $2x_1 + 3x_2 \geq 6$ ④

   $x_1, x_2 \geq 0$



$z = 18 = 6x_1 + 9x_2$ anywhere on edge A.

Here, the slopes of objective fct. and constraint 4 are the same.

$\Rightarrow$ Any point on edge A is an optimal solution, i.e., there are infinitely many.

We call such problems "degenerate".

meaning infinitely many points on the bounded line segment between points $(0,2)$ and $(3,0)$ (i.e., edge A)

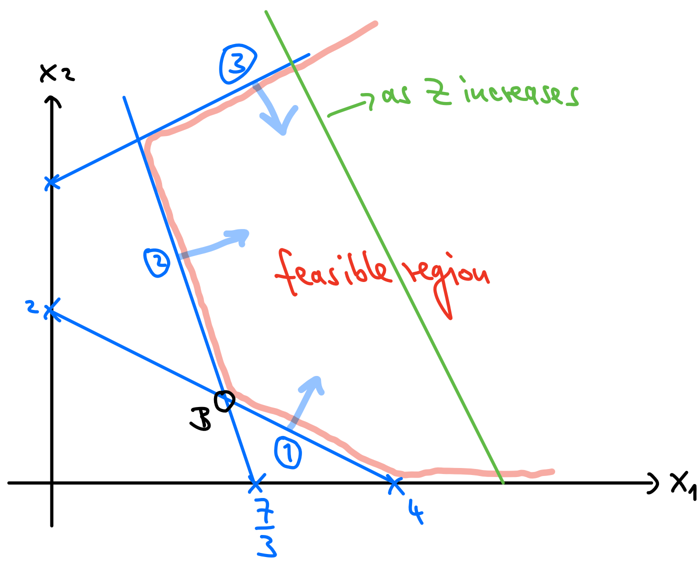
3. • maximize $z = 4x_1 + 2x_2$

   • constraints: $x_1 + 2x_2 \geq 4$ ①

   $3x_1 + x_2 \geq 7$ ②

   $-x_1 + 2x_2 \leq 7$ ③

   $x_1, x_2 \geq 0$



Here, feasible region is unbounded, and Z increases in unbounded direction.
There are (infinitely) many feasible solutions, but none of them is optimal.

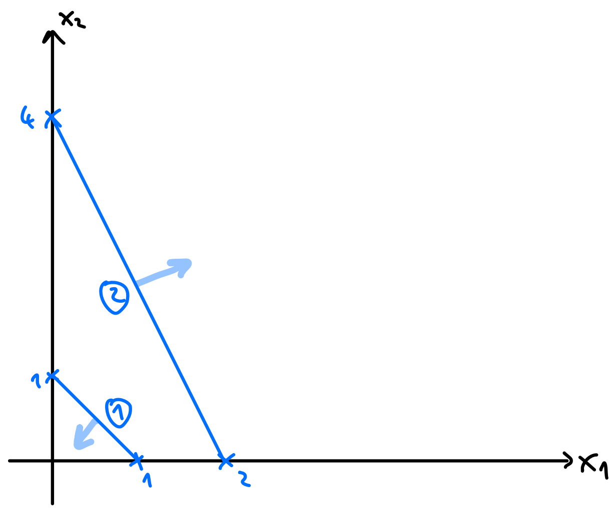
(Note: If Z would be minimized, the optimal solution would be at $B = \binom{2}{1}$, and $Z = 10$.)

4. • maximize $Z = 3x_1 + 4x_2$

• constraints: $x_1 + x_2 \leq 1$ (1)

$2x_1 + x_2 \geq 4$ (2)

$x_1, x_2 \geq 0$



=> The feasible region is empty; there are no feasible solutions.

We call such problems $\boxed{\text{"over-constrained"}}$.

Summary:

The feasible region can be

a) bounded
- either → optimal sol. is $\overbrace{\text{CPF}}^{\text{a cornerpoint}}$ only
- or → ∞ many optimal solutions

b) unbounded
- either → optimal sol. is CPF only
- or → ∞ many optimal solutions
- or → no optimal solution

c) empty → no optimal solution

More generally, two prototypical examples (but mixtures are also possible) of linear Programming (LP) models are:

I) Activity analysis problem: (e.g., Wyndor)

- A = set of activities (or products)
- R = set of resources (or production facilities)
- w_{ij} = workload required from activity $i \in A$ on resource $j \in R$
- c_j = available capacity of resource $j \in R$
- p_i = profit from performing one unit of activity $i \in A$
- decision variables x_i : # of units of activity $i \in A$ to perform

LP problem: • maximize $Z = \sum_{i \in A} p_i x_i$ (total profit)

• constraints: $\sum_{i \in A} w_{ij} x_i \leq c_j$ for all $j \in R$, and $x_i \geq 0$ for all $i \in A$

II) Diet-type problem:

- F = set of foods
- N = set of nutrients
- c_i = unit cost of food $i \in F$
- r_j = minimum requirement for nutrient $j \in N$
- a_{ij} = amount of nutrient $j \in N$ from eating one unit of food $i \in F$
- decision variables x_i = # of units of food $i \in F$ to consume

LP problem: • minimize $Z = \sum_{i \in F} c_i x_i$ (total cost)

• constraint: $\sum_{i \in F} a_{ij} x_i \geq r_j$ for all $j \in N$, and $x_i \geq 0$ for all $i \in F$

2.2 Standard Form of LP Problems

Goal: Bring all LP problems into a standardized form. Then later we can easier develop a general algorithm to solve them.

Goal: Write LP problems in the following $\boxed{\text{standard form}}$: (note: some books might use other very similar standards)

- Minimize $z = c^T x$, with $c \in \mathbb{R}^m$, $x \in \mathbb{R}^m$
- Constraints: $Ax = b$, with A an $n \times m$ matrix, $b \in \mathbb{R}^n$ and $x \geq 0$ (meaning $x_j \geq 0$ for all $j = 1, \dots, m$)

Explanation of notation:

- $c = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ are column vectors

$c^T = (c_1, \dots, c_m) = c$ transpose $=$ row vector

$$\Rightarrow c^T x = (c_1, \dots, c_m) \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \sum_{i=1}^{m} c_i x_i$$

multiplication of a $(1 \times m)$ matrix with an $(m \times 1)$ matrix

- $A = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} = n \times m$ matrix, or $A \in \underbrace{\mathrm{Mat}(n,m)}_{\text{set of } n \times m \text{ matrices}}$

Recall: $$Ax = \begin{pmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} A_{11} x_1 + \dots + A_{1m} x_m \\ A_{21} x_1 + \dots + A_{2m} x_m \\ \vdots \\ A_{n1} x_1 + \dots + A_{nm} x_m \end{pmatrix}$$

i.e., $(Ax)_i = \sum_{j=1}^{m} A_{ij} x_j$

$\Rightarrow Ax = b$ means: $\sum_{j=1}^{m} A_{ij} x_j = b_i$ for all $i = 1, \dots, n$

Claim: $\boxed{\text{Every LP problem can be written in standard form}}$.

We illustrate this with the following example ("proof by example") next time:

- Maximize $z = x_1 + 2x_2 + 3x_3$
- Constraints: $x_1 + x_2 - x_3 = 1$

$$-2x_1 + x_2 + 2x_3 \geq -5$$

$$x_1 - x_2 \leq 4$$

$$x_2 + x_3 \leq 5$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$