Operations Research, Jacobs University, Fall 2022 Session 5

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Problem from last time continued:

Find all solutions of $Ax=b$ for A a wide matrix.

Example: $A=\begin{pmatrix} 1 & 3 & 1 & 1 \\ 2 & 6 & 0 & -1 \end{pmatrix}$, $b=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Last time: We reduced the augmented matrix $\left(\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 2 \\ 2 & 6 & 0 & -1 & 1 \end{array}\right)$ to reduced row echelon form $\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \end{array}\right)$ by using Gaussian elimination.

pivots

$\Rightarrow$ This corresponds to the two equations:

$$x_1 + 3x_2 \quad - \tfrac{1}{2} x_4 = \tfrac{1}{2}$$
$$x_3 + \tfrac{3}{2} x_4 = \tfrac{3}{2}$$

$\Rightarrow$ We have two "free" variables, e.g., $x_4 = -\mu$, $x_2 = -\lambda$

$\hookrightarrow$ 2 (linearly independent) eq.s for 4 variables, i.e., $4-2=2$ variables can be chosen freely

$\Rightarrow$ $x_3 = \frac{3}{2} + \frac{3}{2}\mu$, $x_1 = \frac{1}{2} + 3\lambda - \frac{1}{2}\mu$

The solution to $Ax=b$ in this example is:

$$\Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 3\lambda - \frac{1}{2}\mu \\ -\lambda \\ \frac{3}{2} + \frac{3}{2}\mu \\ -\mu \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{3}{2} \\ 0 \end{pmatrix}}_{\text{called basic solution}} + \underbrace{\lambda \begin{pmatrix} 3 \\ -1 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -\frac{1}{2} \\ 0 \\ \frac{3}{2} \\ -1 \end{pmatrix}}_{\text{the two vectors span the space of solutions to the homogeneous equation } Ax=0}$$

$\left(\Rightarrow x = x^{basic} + x^{hom}, \text{ where } x^{hom} \text{ solves } Ax^{hom} = 0 \right)$

We can use the following trick to immediately read off the solution:

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} \\ \leftarrow \text{add zero row} \\ \\ \leftarrow \text{add zero row} \end{array}$$

replace 0's on diagonal by -1

$$\Rightarrow \text{solution} = \begin{pmatrix} \\ \\ \end{pmatrix} + \lambda \begin{pmatrix} \\ \\ \end{pmatrix} + \mu \begin{pmatrix} \\ \\ \end{pmatrix}$$

Recipe to get solutions directly from augmented matrix in row echelon form:

- add zero rows s.t. pivots, i.e., leading 1's, are on diagonal
- put -1 on diagonal in the zero rows
- read off solution as above

We call $\mathcal{B}$ the set of basic solution columns, i.e., the set containing the index of each column with a pivot. Here, $\mathcal{B} = \{1, 3\}$.
For the basic solution x^{basic} we have $x_j^{\text{basic}} = 0$ for $j \notin \mathcal{B}$.

But note: there are many ways to parametrize the solutions, e.g., also:

$$\begin{array}{c} \text{pivot columns} \\ \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 & -1 & 0 \end{array}\right) \end{array}$$

$(R_1 \text{ above}/3)$ i.e., $\mathcal{B} = \{2, 3\}$

$$\Rightarrow x = \begin{pmatrix} 0 \\ \frac{1}{6} \\ \frac{3}{2} \\ 0 \end{pmatrix} + \tilde{\lambda} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\mu} \begin{pmatrix} 0 \\ -\frac{1}{6} \\ \frac{3}{2} \\ -1 \end{pmatrix}$$

Another possibility: $\left(\begin{array}{cccc|c} \frac{1}{3} & 1 & 0 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right)$

$\frac{1}{6}R2 + R1 \to R1$ $\left(\begin{array}{cccc|c} \frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right)$

$\Rightarrow B = \{2, 4\}$, read off solution:

$$\begin{array}{c} \begin{array}{ccccc} \text{put } -1 & & \text{put } -1 & & \end{array} \\ \left(\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & \frac{1}{9} & 0 & \frac{1}{3} \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 & 1 \end{array}\right) \end{array}$$

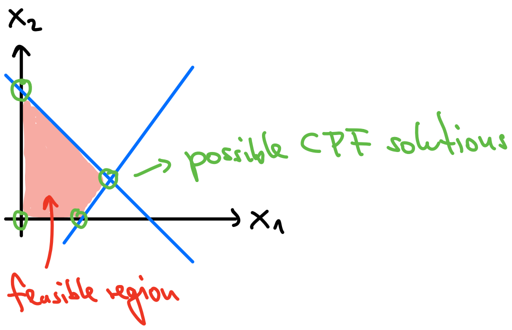
$$\Rightarrow \quad x = \begin{pmatrix} 0 \\ \frac{1}{3} \\ 0 \\ 1 \end{pmatrix} + \tilde{\tilde{\lambda}} \begin{pmatrix} -1 \\ \frac{1}{3} \\ 0 \\ 0 \end{pmatrix} + \tilde{\tilde{\mu}} \begin{pmatrix} 0 \\ \frac{1}{9} \\ -1 \\ \frac{2}{3} \end{pmatrix}$$

Conclusion: We know how to compute all solutions to $Ax = b$, A a wide matrix, in the specific form $x = x^{basic} + x^{hom}$, where x^{hom} solves $Ax^{hom} = 0$, and x^{basic} has at least as many 0 entries as "number of columns" minus "number of pivots".

Now back to the standard form of LP problems:
- minimize $z = c^T x$
- constraints: $Ax = b$ and $x \geq 0$

How does the feasible region look like?

• In our first examples (not standard form): polygon



Intuition: if there is an optimal solution, there is at least one which is $\underbrace{\text{CPF}}$.

= "Corner Point Feasible", i.e., a solution that is at a corner point of the feasible region

• In standard form? Next time!