Operations Research, Jacobs University, Fall 2022 — Session 6

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Conclusion from last time:

We know how to compute all solutions to $Ax=b$, A a wide matrix, in the specific form $x = x^{basic} + x^{hom}$, where x^{hom} solves $Ax^{hom}=0$, and x^{basic} has at least as many 0 entries as $\underbrace{\text{"number of columns"}}_{\text{= total number of variables in standard form LP problem, i.e., } x_1, x_2, \dots, s_1, s_2, \dots, u, v, \dots}$ minus "number of pivots".

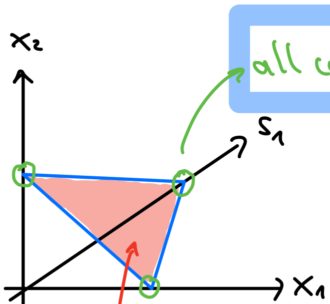
Now back to the standard form of LP problems:
- minimize $z = c^T x$
- constraints: $Ax = b$ and $x \geq 0$

How does the feasible region look like?

$Ax=b$ describes an affine subspace, e.g., a plane in 3d.

The feasible region is that part of the subspace with $\underbrace{x \geq 0}_{\text{= "quadrant" where all components are nonnegative (i.e., positive or zero)}}$!

E.g.:



$\boxed{\text{all corner points are basic feasible solutions}}$

feasible region has shape of a "simplex"

Important insight: If there is an optimal solution, we can always find one at a cornerpoint. And the cornerpoints correspond to the basic solutions of $Ax=b$. These we can find with Gaussian elimination.

A feasible region like in the picture could arise from, e.g., $A = \underbrace{(5,3,4)}_{1\times 3 \text{ matrix}}$ and $b = \underbrace{2}_{\text{vector with 1 component}}$.

$$\Rightarrow \text{augmented matrix: } \begin{array}{cccc} x_1 & x_2 & s_1 & \\ \downarrow & \downarrow & \downarrow & \\ \end{array}$$
$$\left(\begin{array}{ccc|c} 5 & 3 & 4 & 2 \end{array}\right)$$

The possibilities for basic solutions are:

- pivot in column 1: $R_1/5 \Rightarrow \left(\begin{array}{ccc|c} 1 & \frac{3}{5} & \frac{4}{5} & \frac{2}{5} \end{array}\right)$

$$\Rightarrow x_1^{\text{basic}} = \begin{pmatrix} 2/5 \\ 0 \\ 0 \end{pmatrix}$$

- pivot in column 2: $R_1/3 \Rightarrow \left(\begin{array}{ccc|c} \frac{5}{3} & 1 & \frac{4}{3} & \frac{2}{3} \end{array}\right)$

$$\Rightarrow x_2^{\text{basic}} = \begin{pmatrix} 0 \\ 2/3 \\ 0 \end{pmatrix}$$

- pivot in column 3: $R_1/4 \Rightarrow \left(\begin{array}{ccc|c} \frac{5}{4} & \frac{3}{4} & 1 & \frac{1}{2} \end{array}\right)$

$$\Rightarrow x_3^{\text{basic}} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Now suppose $c = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Then we check: $z_1 = c^T x_1^{\text{basic}} = (1,2,1) \begin{pmatrix} 2/5 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \frac{2}{5} + 2 \cdot 0 + 1 \cdot 0 = \frac{2}{5}$

$$z_2 = c^T x_2^{\text{basic}} = (1,2,1) \begin{pmatrix} 0 \\ \frac{2}{3} \\ 0 \end{pmatrix} = 1 \cdot 0 + 2 \cdot \frac{2}{3} + 1 \cdot 0 = \frac{4}{3}$$

$$z_3 = c^T x_3^{\text{basic}} = (1,2,1) \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix} = 1 \cdot 0 + 2 \cdot 0 + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$\Rightarrow z_1$ is the smallest, i.e., x_1^{basic} is our optimal solution!

Let us write down our main insight more precisely:

Theorem: If a standard form LP problem has optimal solutions, then there is an optimal basic solution (i.e., an optimal solution that is a vertex/corner of the feasible region).

Proof idea:

Suppose x is optimal, but not a cornerpoint. Then there is always a vector v such that both $x+v$ and $x-v$ are still feasible. In fact, since x minimizes $c^T x$:

$$\left.\begin{array}{l} \cdot\ c^T x \le c^T(x+v) \Rightarrow c^T v \ge 0 \\ \cdot\ c^T x \le c^T(x-v) \Rightarrow c^T v \le 0 \end{array}\right\} \Rightarrow c^T v = 0,$$

so $x+v$ and $x-v$ are also optimal!

Then: go along either v or $-v$ until we hit the boundary of the feasible region, i.e., until one component becomes negative.

$\Rightarrow$ Repeat until we get stuck at a corner (where either v or $-v$ will lead out of the feasible region). $\square$