2.3 The Simplex Method

Conclusion from previous chapter:

- For a standard form LP problem we need to check only basic $\underbrace{\text{feasible}}$ solutions.
  all components ≥ 0
- We can find basic solutions with Gaussian elimination.

So in order to find optimal solutions, we could simply go through all possible basic feasible solution.

BUT: For large problems, this is way too computationally expensive!

The following method is faster:

Simplex algorithm:

(i) Start with any basic feasible solution.

(ii) Swap one basic variable ("leaving variable") for another variable ("entering variable") s.t. objective function improves the most.

(iii) Repeat this until no improvement of objective fct. is possible.

Let us work out the details using the example from Session 4.

We write it as a "simplex tableau":

$$
\begin{array}{c} \\ A \\ \\ \\ \\ c^T \end{array}
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
2 & -1 & -2 & 2 & 1 & 0 & 0 & 5 \\
1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
-1 & -2 & -3 & 3 & 0 & 0 & 0 & z
\end{array}
\begin{array}{c} \\ \\ b \\ \\ \\ \\ \end{array}
$$

(i) We need to find a basic solution.

Let us choose x_1, s_1, s_2, s_3 columns as our pivot columns.

$\Rightarrow$ We need to eliminate ○ entries.

now these four columns are the pivot columns

$$
\Rightarrow \quad
\begin{array}{r}
\\
\\
-2R_1+R_2 \to R_2: \\
-R_1+R_3 \to R_3: \\
\\
R_1+R_5 \to R_5:
\end{array}
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
0 & -1 & -4 & 4 & 0 & 0 & 0 & z+1
\end{array}
$$

Note: By eliminating the "pivot column entry" here, we can read off z on the right ($z=-1$ here).

meaning all components ≥ 0

A basic feasible solution is: $x_1 = 1, x_2 = 0, u = 0, v = 0, s_1 = 3, s_2 = 3, s_3 = 5$.

recall how we got basic solutions (Sessions 4 and 5)

$$\tilde{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 3 \\ 3 \\ 5 \end{pmatrix}$$

There, $c^T x = 0 = z + 1$ i.e., $z = -1$.

c^T is the last row (left-hand side)

Note: We make one more simplification in the notation:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
0 & -1 & -4 & 4 & 0 & 0 & 0 & 1
\end{array}
$$

we delete the z here; then this number is equal to $-z$ at the basic solution.

this tracks $-z$, i.e., this shall be maximized

(ii) Entry variable: Go along direction that improves objective function the most.

new entry variable

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
0 & -1 & -4 & 4 & 0 & 0 & 0 & 1
\end{array}
$$

Geometrically: Go along direction where the slope is most negative.

choose the column variable where the entry in the last row is the most negative. (If all entries are positive, we are done, because z cannot be decreased further.)

Leaving variable: Where we put the new pivot.

where the pivot in R_4 used to be

Let's test: Take pivot (in column u) in R_4 (row 4), i.e., s_3 as leaving variable

needs to be zero

$$
\begin{array}{lccccccc|c}
R_3: & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
R_4: & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5
\end{array}
\qquad
\begin{array}{lccccccc|c}
R_3 - R_4 \to R_3: & 0 & -3 & 0 & 0 & 0 & 1 & -1 & -2 \\
 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 5
\end{array}
$$

does not work, because here $s_2 = -2$

(but we need $s_2 \geq 0$ to be feasible)

note: $x_2 = v = s_3 = 0$ at new basic sol.

$\Rightarrow$ Need to choose another leaving variable (next time).