We continue our example from last time:

new entry variable

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\
0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 & 5 \\
\hline
0 & -1 & -4 & 4 & 0 & 0 & 0 & 1
\end{array}
$$

here z decreases the most

Question: What is the leaving variable?

Last time: pivot in R_4 (s_3 as leaving variable) did not work, since then s_2 became negative.

Why did this happen?

↳ At basic solution, with including u (as the entering variable), we have:

$$x_1 = 1 - (-1)u \geq 0 \implies \text{no bound on } u$$

$$s_1 = 3 - 0 \cdot u \geq 0 \implies \text{no bound on } u$$

$$s_2 = 3 - 1 \cdot u \geq 0 \implies \text{need } u \leq \tfrac{3}{1}$$

$$s_3 = 5 - 1 \cdot u \geq 0 \implies \text{need } u \leq \tfrac{5}{1}$$

→ so if we increase u up to 5 we will violate this constraint

How do x_1, s_1, s_2, s_3 change if we introduce some (small) positive u?

General rule: take the row with the least positive ratio of coefficient from right-most column to coefficient in new entry variable column.

$\implies$ Here, we need to take pivot in R_3, i.e., s_2 as leaving variable

We compute:

$$
\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
R_3+R_1\to R_1: & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 4 \\
 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 3 \\
 & 0 & -2 & 1 & -1 & 0 & 1 & 0 & 3 \\
R_4-R_3\to R_4: & 0 & 3 & 0 & 0 & 0 & -1 & 1 & 2 \\
\hline
4R_3+R_5\to R_5: & 0 & -9 & 0 & 0 & 0 & 4 & 0 & 13
\end{array}
$$

$\Rightarrow x_1=4, u=3, s_1=3, s_3=2,$

$x_2=0, v=0, s_2=0$

and $z=-13$

next entry variable: x_2

new pivot (only positive entry in this column) $\Rightarrow$ new leaving variable: s_3

(iii) Repeat: entry variable x_2, leaving variable s_3

$$
\begin{array}{r|ccccccc|c}
 & x_1 & x_2 & u & v & s_1 & s_2 & s_3 & \\
\hline
\frac{1}{3}R_4+R_1\to R_1: & 1 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{14}{3} \\
R_4+R_2\to R_2: & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 5 \\
\frac{2}{3}R_4+R_3\to R_3: & 0 & 0 & 1 & -1 & 0 & \frac{1}{3} & \frac{2}{3} & 13/3 \\
\frac{1}{3}R_4\to R_4: & 0 & 1 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{3} & \frac{2}{3} \\
\hline
3R_4+R_5\to R_5: & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 19
\end{array}
$$

$\Rightarrow$ $\boxed{x_1=\frac{14}{3},\ x_2=\frac{2}{3},\ u=\frac{13}{3},\ s_1=5,\ v=0,\ s_2=0,\ s_3=0, \text{ and } z=-19}$

both are non-negative

$\Rightarrow$ no further improvement possible

$\Rightarrow$ We found an optimal basic solution.

Summary of rules:

- Entering variable column: choose the one with most negative entry in objection function row.

  If all entries are non-negative, solution has been found (terminate the algorithm).

- Leaving variable: choose row with the least positive ratio of right-hand coefficient to coefficient in that column.
  If not possible (if all coefficients in column are negative), we have found that the feasible region is unbounded and objective function can be made arbitrarily small.

Note: sometimes it is not easy to choose a feasible basic solution to start with; we will deal with that later.

Another example:

Maximize $z = 2x_1 + x_2$ with constraints

$$-x_1 + x_2 \le 1 \quad \Rightarrow \quad -x_1 + x_2 + s_1 = 1$$
$$x_1 - 2x_2 \le 2 \qquad\quad x_1 - 2x_2 + s_2 = 2$$
$$x_1, x_2 \ge 0$$

Simplex tableau:

new entry row

x_1	x_2	s_1	s_2	
-1	1	1	0	1
1	-2	0	1	2
-2	-1	0	0	0

both positive, so we can immediately see that a basic feasible solution is $x_1 = 0, x_2 = 0, s_1 = 1, s_2 = 2$ (with $z = 0$).

least positive ratio (actually the only positive ratio here)

$\Rightarrow s_2 =$ leaving variable

	x_1	x_2	s_1	s_2	
$R_2 + R_1 \to R_1$:	0	-1	1	1	3
	1	-2	0	1	2
$2R_2 + R_3 \to R_3$:	0	-5	0	2	4

$\hookrightarrow x_2$ should be new entry variable

but none of these ratios are positive!

$\Rightarrow$ We can increase x_2 as much as we like (no boundary constraint), i.e., we can make z more negative without bounds.

Graphically: