We continue our example from last time:

Maximize $z = 3x_1 + 2x_2$, subject to the constraints:

$$5x_1 \leq 100 \quad (1)$$
$$10x_2 \leq 100 \quad (2)$$
$$4x_1 + 3x_2 \leq 100 \quad (3)$$
$$3x_1 + 5x_2 \leq 100 \quad (4)$$
$$x_1, x_2 \geq 0$$

With the graphical method (or simplex method), we find that $s_1 = 0 = s_3$, i.e., constraints (1) and (3) are binding (no slack).

So the optimal solution $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ can be found from solving $\tilde{A}x = \tilde{b}$,

with $\tilde{A} = \begin{array}{cc} x_1 & x_2 \end{array}\begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}$ $\begin{array}{l} \leftarrow \text{constraint (1)} \\ \leftarrow \text{constraint (3)} \end{array}$, $\tilde{b} = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$.

$$\Rightarrow x = \tilde{A}^{-1}\tilde{b} \quad \text{and} \quad z = c^T x = \underbrace{c^T \tilde{A}^{-1}}_{=:\tilde{y}} \tilde{b}, \text{ where we define } \tilde{y} = c^T \tilde{A}^{-1},$$

and more generally, $y = \begin{pmatrix} \tilde{y}_1 \\ 0 \\ \tilde{y}_2 \\ 0 \end{pmatrix}$ ← putting 0 at the non-binding constraints; here: (2) and (4)

Here concretely: $\tilde{y}^T = c^T \tilde{A}^{-1} = (3, 2) \begin{pmatrix} 5 & 0 \\ 4 & 3 \end{pmatrix}^{-1} = (3, 2) \frac{1}{15} \begin{pmatrix} 3 & 0 \\ -4 & 5 \end{pmatrix}$

$$= \frac{1}{15}(1, 10). \quad \left(\Rightarrow y^T = \frac{1}{15}(1, 0, 10, 0) \right)$$

Now: Change capacities b by a small amount; small meaning the binding constraints remain the same.

Then $b \longrightarrow b+\delta$ and we find $x = \tilde{A}^{-1}(\tilde{b}+\tilde{\delta})$.

$\Rightarrow$ new profit $z(\delta) = y^T(b+\delta) = \underbrace{y^T b}_{=z(0)} + y^T\delta$

The $y_1, \dots, y_m$ are called $\boxed{\text{shadow prices.}}$ These are the changes of profit per unit of capacity at current operating conditions.

Indeed, the following holds:

> <u>Theorem:</u> The value of a company in terms of maximal profit from its operation equals the value of all its resources valued at the current shadow prices.

Proof: Set $\delta = -b$, so $z(\delta) = y^T(b-b) = 0$ (no resources, no operations, no profit)

↳ Note: this does not change the binding constraints (even though $\delta=-b$ is large), since we just rescale the feasible region proportionally.

Thus $z(\delta) = z(0) - y^T b = 0$, so $\underbrace{z(0)}_{\text{value of the company}} = \underbrace{y^T b}_{\text{resources valued at shadow prices}}$. $\square$

In our example, we found $y = \begin{pmatrix} \frac{1}{15} \\ 0 \\ \frac{10}{15} \\ 0 \end{pmatrix}$. So for example, increasing the capacity of constraint (1) by one unit will increase the profit by $\frac{1}{15}$.

If we could choose to increase the working hours for either constraint, we should choose constraint (3) because this increases the profit the most.

Next: How to compute shadow prices directly (via solving the "dual" LP problem).

Recall the example: maximize profit $z = 3x_1 + 2x_2 = c^T x$

with constraints
$$\left.\begin{aligned} 5x_1 &\le 100 \\ 10x_2 &\le 100 \\ 4x_1 + 3x_2 &\le 100 \\ 3x_1 + 5x_2 &\le 100 \end{aligned}\right\} Ax \le b$$
$$x_1, x_2 \ge 0$$

Now: Consider the following scenario: A company wants to buy our production capacity.

What are fair prices y_1, y_2, y_3, y_4 for the resources (1), (2), (3), (4)?

In our example:
- profit per car: 3
- profit per truck: 2
- current car assembly hours: 5 for constraint (1), 4 for constraint (3), 3 for constraint (4)
- trucks: 10 for (2), 3 for (3), 5 for (4)

Thus we want:
$$\left.\begin{aligned} &\cdot\ 5y_1 + 4y_3 + 3y_4 \ge 3 \\ &\cdot\ \underbrace{10y_2 + 3y_3 + 5y_4}_{= A^T y} \ge \underbrace{2}_{c} \end{aligned}\right\}$$
selling capacity to produce one car/truck needs to be at least as profitable as producing a car/truck

(Recall: $A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \\ 4 & 3 \\ 3 & 5 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 5 & 0 & 4 & 3 \\ 0 & 10 & 3 & 5 \end{pmatrix}$ is the transpose of A.)

The price for all capacity is $y_1 \cdot 100 + \dots + y_4 \cdot 100 = b^T y$. Minimizing this yields the minimum prize.

This leads to the "dual problem": • minimize $b^T y$,
• subject to $A^T y \geq c$ and $y \geq 0$.

as compard to the original "primal problem": • maximize $c^T x$,
• subject to $Ax \leq b$ and $x \geq 0$.

Solving the dual problem gives us the shadow prices.

Two results about the relation between dual and primal LP:

• Note that $c^T x = x^T c \underset{c \leq A^T y}{\leq} x^T A^T y \underset{(AB)^T = B^T A^T}{=} (Ax)^T y \underset{Ax \leq b}{\leq} b^T y$.

This is known as weak duality:

If x is a solution to the primal problem (i.e., x is feasible, but not necessarily optimal), and y is a solution to the dual problem, then $c^T x \leq b^T y$.

• A bit harder to prove (but intuitively clear) is strong duality:

The dual has an optimal solution if and only if the primal does. In this case $c^T x = b^T y$.