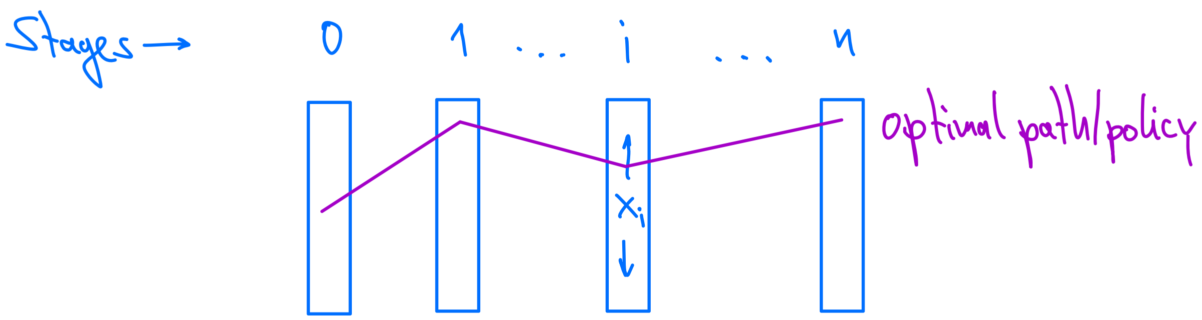
3. Further Optimization Techniques

3.1 Dynamic Programming

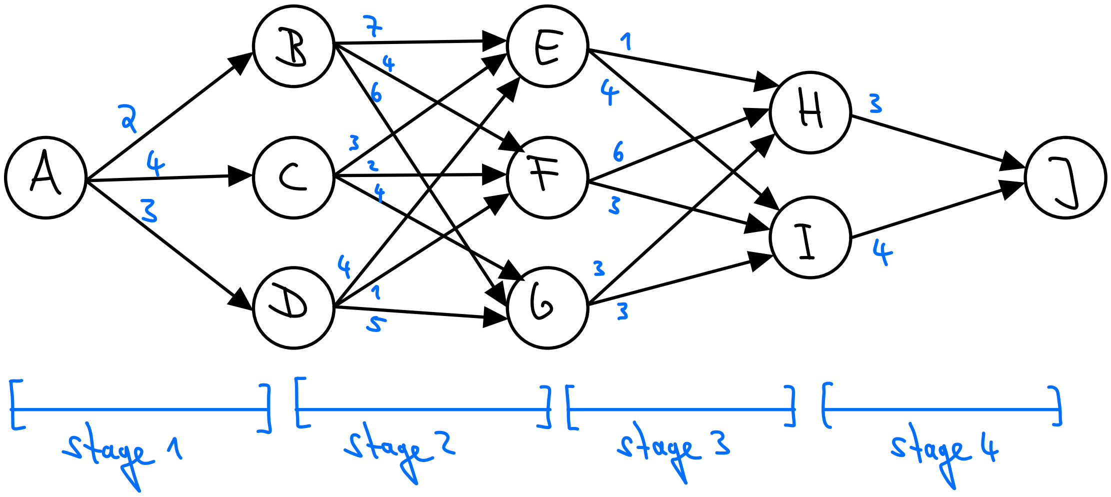
Setting of Dynamic Programming:

- problem can be divided into several "stages"
- a "policy decision" (= transition from one "state" to another) needs to be made at each stage
- goal: find optimal sequence of decisions (= "optimal policy") ← usually a min. or max. problem
- decision variables: x_i = state to transition to in stage i (from some state at stage $i-1$).



Example: Stagecoach problem (Hillier, Lieberman: Chapter 10.1)

Need to travel from A to J; travel/insurance costs are associated to different route segments:



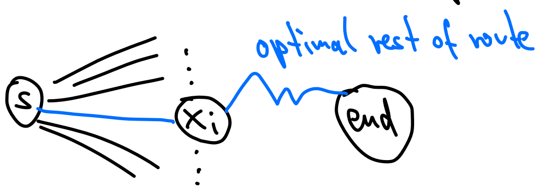
Note: This is a special type of shortest path problem: one with different stages.

Route: $A \to x_1 \to x_2 \to x_3 \to x_4 = J$

Solution procedure: Start from the end and go backwards through the stages.

We introduce:

- $f_i(s, x_i)$ = cost of optimal travel route starting at s at stage $i-1$, passing through x_i at stage i, going to the end

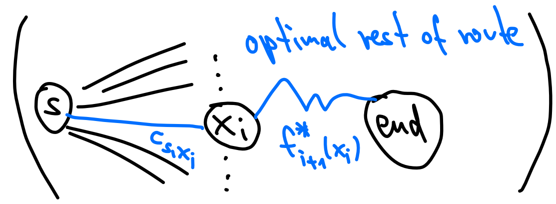


- $f_i^*(s) = \min_{x_i} f_i(s, x_i)$ = cost of optimal travel route starting at s

  Let x_i^* denote the minimum (not necessarily unique) = optimal choice at stage i

- $\Rightarrow$ In our example:

$$f_i(s, x_i) = \underbrace{c_{s,x_i}}_{\text{cost of travel from } s \text{ to } x_i\text{; see network above (given parameters)}} + \underbrace{f_{i+1}^*(x_i)}_{\text{optimal future cost to travel from } x_i \text{ to the end}}$$



Solution:

stage $i=4$:

s	f_4^*	x_4^*
H	3	J
I	4	J

$i=3$:

	$f_3(s,x_3)=c_{s,x_3}+f_4^*(x_3)$			
s	$x_3=H$	$x_3=I$	$f_3^*(s)$	x_3^*
E	$\underbrace{1}_{c_{E,H}}+\underbrace{3}_{f_4^*(H)}=4$	$4+4=8$	4	H
F	$6+3=9$	$3+4=7$	7	I
G	$3+3=6$	$3+4=7$	6	H

$i=2$

	$f_2(s,x_2)=c_{s,x_2}+f_3^*(x_2)$				
s	$x_2=E$	$x_2=F$	$x_2=G$	$f_2^*(s)$	x_2^*
B	$7+4=11$	$4+7=11$	$6+6=12$	11	E or F
C	$3+4=7$	$2+7=9$	$4+6=10$	7	E
D	$4+4=8$	$1+7=8$	$5+6=11$	8	E or F

$i=1$

	$f_1(s,x_1)=c_{s,x_1}+f_2^*(x_1)$				
s	$x_1=B$	$x_1=C$	$x_1=D$	$f_1^*(s)$	x_1^*
A	$2+11=13$	$4+7=11$	$3+8=11$	11	C or D

$\Rightarrow$ Minimal cost is 11, optimal paths are

A–C–E–H–J or A–D–E–H–J or A–D–F–I–J