Next, let us consider stochastic models for perishable products, also called "newsvendor problem".

We consider/assume:

- A single perishable product, e.g., newspaper, food, flowers, seasonal goods such as clothing (but also, e.g., airline reservations)
- single time period
- at the end of period, product has salvage value (e.g., selling clothes out of season at a discount)
- no initial inventory
- decision variable y = # of items to stock
- the demand D is a random variable (we will need to make reasonable assumptions for its probability distribution)

- K = setup cost, irrelevant here (exactly one order is placed)
- c = unit cost of purchasing/producing
- h = holding cost per item = cost of storage − salvage value
- p = shortage cost (penalty) per item, e.g., lost revenue or lost customer goodwill

The amount sold is $\min\{D, y\} = \begin{cases} D & \text{if } D < y, \\ y & \text{if } D \geq y. \end{cases}$

The cost is $C(D,y) = \underbrace{cy}_{\text{order cost}} + \underbrace{p\max\{0, D-y\}}_{\substack{\text{penalty} \\ = \begin{cases} 0 & \text{if } D<y, \\ p(D-y) & \text{if } D>y. \end{cases}}} + \underbrace{h\max\{0, y-D\}}_{\substack{\text{holding cost} \\ = \begin{cases} 0 & \text{if } D>y \text{ (all sold)} \\ h(y-D) & \text{if } D<y \text{ (leftovers)} \end{cases}}}$

Goal: minimize expected cost, given some probability distribution $\underbrace{P_D(d)}_{\text{probability that demand} = d}$ for the demand.

Expected cost $\mathbb{E}[C](y) = \sum_{d=0}^{\infty} C(d,y)\, P_D(d)$.

How do we model $P_D(d)$?

Possibility (1): brute-force using empirical data i.e., P_D = empirical probability distribution

Problems:
- might not have enough data (e.g., certain numbers of items never sold)
- historical data not always good

Possibility (2): use a theoretic $P_D(d)$, using additionally mean or spread of historical data

If d ranges over large number of values, it makes sense to approximate it by a continuous probability distribution $\varphi(d)$.

Then
$$\mathbb{E}[C](y) = \int_0^\infty C(x,y)\,\varphi(x)\,dx$$
$$= \int_0^\infty \left[cy + p\max\{0, x-y\} + h\max\{0, y-x\} \right] \varphi(x)\,dx$$
$$= cy \underbrace{\int_0^\infty \varphi(x)dx}_{\substack{=1, \text{ since} \\ \varphi \text{ is a probability} \\ \text{distribution}}} + p \underbrace{\int_0^\infty \max\{0, x-y\}\varphi(x)dx}_{= \int_y^\infty (x-y)\varphi(x)dx} + h \underbrace{\int_0^\infty \max\{0, y-x\}\varphi(x)dx}_{= \int_0^y (y-x)\varphi(x)dx}.$$

$$\Rightarrow \mathbb{E}[C](y) = cy + p\int_y^\infty (x-y)\varphi(x)dx + h\int_0^y (y-x)\varphi(x)dx$$

Goal: minimize $\mathbb{E}[C](y)$. Thus we compute:

$$\frac{d\,\mathbb{E}[C](y)}{dy} = c + \underbrace{p\int_y^\infty (-\varphi(x))dx - p(x-y)\varphi(x)\Big|_{x=y}} + h\int_0^y \varphi(x)dx + h(y-x)\varphi(x)\Big|_{x=y}$$

Note: $\frac{d}{dy}\int_y^\infty f(x,y)dx = \frac{d}{dy}\left[F(\infty,y) - \underbrace{F(y,y)}_{=F(x_1,x_2)|_{x_1=x_2=y}}\right]$ (F anti-derivative in first variable, i.e., $\frac{\partial}{\partial x}F(x,y) = f(x,y)$)

$$= \frac{\partial}{\partial y}F(\infty,y) - \underbrace{\frac{\partial}{\partial x_1}F(x_1,y)\Big|_{x_1=y}}_{=f(y,y)} - \frac{\partial}{\partial x_2}F(y,x_2)\Big|_{x_2=y}$$

$$= \int_y^\infty \frac{\partial}{\partial y}f(x,y)dx - f(y,y)$$

$$\Rightarrow \frac{d\,\mathbb{E}[C](y)}{dy} = c - p\int_y^\infty \varphi(x)dx + h\int_0^y \varphi(x)dx$$

Let us introduce the cumulative distribution function $\Phi(y) := \int_0^y \varphi(x)dx$.

Note that $\Phi(\infty) = \int_0^\infty \varphi(x)dx = 1$ (all probabilities integrate up to 1).

$\Phi(y)$ tells us the probability that the demand is satified if we order y items.

Then $$\frac{d\,\mathbb{E}[C](y)}{dy} = c - p\Big(\underbrace{\int_0^\infty \varphi(x)dx}_{=1} - \underbrace{\int_0^y \varphi(x)dx}_{=\Phi(y)}\Big) + h\underbrace{\int_0^y \varphi(x)dx}_{=\Phi(y)}$$

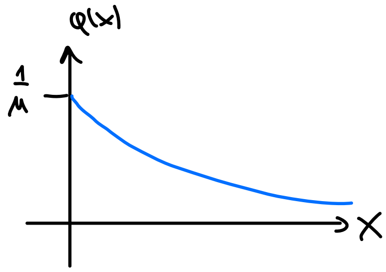
$$= c - p + (p+h)\Phi(y) \overset{!}{=} 0$$

$$\Rightarrow \boxed{\Phi(y^*) = \frac{p-c}{p+h}}$$, i.e., we should choose y^* s.t. this equation is satified.

solutions can be found algebraically or numerically / graphically

$\Phi(y^*)$ is called "$\underbrace{\text{optimal service level}}_{\text{= probability that demand is satisfied}}$".

Example: Assume exponential distribution $\varphi(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}}$, with $\mu > 0$ the mean value.



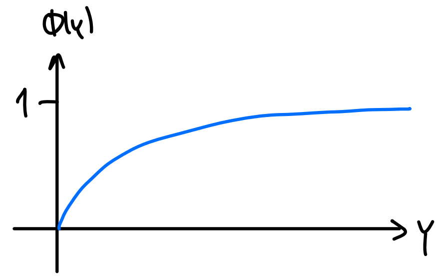
Note: μ is indeed the mean, since

change of variables: $\frac{x}{\mu} = y$

$$\mathbb{E}(X) = \int_0^\infty x\,\varphi(x)\,dx = \int_0^\infty \frac{x}{\mu} e^{-\frac{x}{\mu}}\,dx = \mu \int_0^\infty y e^{-y}\,dy$$

integration by parts

$$\overset{\text{integration by parts}}{=} \underbrace{\mu\left[-ye^{-y}\right]_0^\infty}_{=0} + \mu \int_0^\infty e^{-y}\,dy = -\mu e^{-y}\Big|_0^\infty = 0 - (-\mu) = \mu$$



Then $\Phi(y) = \int_0^y \frac{1}{\mu} e^{-\frac{x}{\mu}}\,dx = -e^{-\frac{x}{\mu}}\Big|_0^y = -e^{-\frac{y}{\mu}} + 1$

$$\Phi(y) = \frac{p-c}{p+h} \iff 1 - e^{-\frac{y}{\mu}} = \frac{p-c}{p+h} \iff e^{-\frac{y}{\mu}} = 1 - \frac{(p-c)}{p+h} = \frac{p+h}{p+h} - \frac{(p-c)}{p+h} = \frac{c+h}{p+h}$$

natural logarithm

$\ln\frac{a}{b} = -\ln\frac{b}{a}$

$$\Rightarrow -\frac{y}{\mu} = \ln\frac{c+h}{p+h} \Rightarrow y = -\mu \ln\frac{c+h}{p+h} = \mu \ln\frac{p+h}{c+h}$$

$\Rightarrow$ For exponential distribution with mean μ, the optimal order quantity is $\boxed{y^* = \mu \ln\frac{p+h}{c+h}}$.

Numerical example: For $\mu = 10\,000$, $c = 200$, $p = 450$, $h = -90$, we find

a large salvage value can make h negative.

$y^* \approx 11\,856$ (so 1856 items more than the average should be stocked).

Note that $\underbrace{\Phi(y^*) = \frac{450-200}{450-90}}_{\text{optimal service level}} = 0.694$ i.e., the demand is satisfied with 69.4% probability here.